1.a. (10 points). Determine the rank of the matrix

$$A = \begin{pmatrix} -1 & 1 & -11 & -1 \\ -8 & 1 & -1 & 1 \\ -3 & 0 & 3 & 0 \end{pmatrix}.$$

rank$(A) = 3$.

Reduce and see that there are 3 pivots.

See 308 G's solutions for more details.

The next three problems involve the matrix A from Question 1.a.

1.b. (5 points). What is the rank of A^T?

$\text{rank}(A^T) = \text{rank}(A) = 3$

1.c. (5 points). Is there a solution to the the equation

$$Ax = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}?$$

Yes. The range of A is all of $\mathbb{R}^3$ since $\text{rank}(A) = 3$.

1.d. (5 points). Suppose that $\mathbf{b} \in \mathbf{R}^3$ is a vector such that $A\mathbf{x} = \mathbf{b}$ has a solution. Is this solution unique?

No, there is an independent variable.

2.a. (20 points). Find the inverse of the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 1 & -2 \end{pmatrix}.$$

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -1 & \frac{5}{2} & -\frac{1}{2} \\ -1 & 2 & -1 \end{bmatrix}$$

Details for the method are in Section G's solutions.

2.b. (5 points). Find the unique solution to $A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

$$\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} \\ -1 & \frac{5}{2} & -\frac{1}{2} \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ -2 \end{bmatrix}.$$

3. (20 points). Find a basis for the null space of the matrix

$$\begin{pmatrix} -2 & -8 & 0 & 0 & 2 \\ 0 & -6 & 0 & 1 & 1 \end{pmatrix}.$$

We use that x_1 and x_2 are dependent variables, while x_3, x_4, x_5 are independent.

$$\begin{aligned} -2x_1 - 8x_2 \qquad\qquad + 2x_5 &= 0 \\ -6x_2 \qquad + x_4 + x_5 &= 0 \end{aligned}$$

So, we fill in

$$\begin{bmatrix} ? \\ ? \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} ? \\ ? \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

subject to the constraint $A\vec{x} = \vec{0}$. We get that

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{6} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $N(A)$.

5.a. (20 points). Describe geometrically the column space of the matrix

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 16 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & -2 \end{pmatrix}.$$

$R_1 \leftrightarrow R_4$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & -2 & -2 & b_4 \\ 16 & -1 & 1 & b_2 \\ 0 & 0 & -1 & b_1 \\ -1 & 0 & 1 & b_3 \end{bmatrix}$$

$R_2 - 16R_1$

$R_4 + R_1$

$$\begin{bmatrix} 1 & -2 & -2 & b_4 \\ 0 & 31 & 33 & b_2 - 16b_4 \\ 0 & 0 & -1 & b_1 \\ 0 & -2 & -1 & b_3 + b_4 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & -2 & -2 & b_4 \\ 0 & -2 & -1 & b_3 + b_4 \\ 0 & 0 & -1 & b_1 \\ 0 & 31 & 33 & b_2 - 16b_4 \end{bmatrix}$$

$R_4 + \frac{31}{2} R_2$

$33 - \frac{31}{2}$

$= \frac{66-31}{2}$

$= \frac{35}{2}$

$$\begin{bmatrix} 1 & -2 & -2 & b_4 \\ 0 & -2 & -1 & b_3 + b_4 \\ 0 & 0 & -1 & b_1 \\ 0 & 0 & \frac{35}{2} & b_2 - 16b_4 + \frac{31}{2}(b_3 + b_4) \end{bmatrix}$$

$R_4 + \frac{35}{2} R_1$

$$\begin{bmatrix} 0 & 0 & 0 & b_2 - 16b_4 + \frac{31}{2}(b_3 + b_4) + \frac{35}{2} b_1 \end{bmatrix}$$

$$\boxed{\frac{35}{2} b_1 + b_2 + \frac{31}{2} b_3 - \frac{1}{2} b_4 = 0}$$

5.b. (10 points) Circle the dimension of the null space of A: $\boxed{0}$ 1, 2, 3, 4, 5.

Use rank + nullity = 3.