308G - Quiz 3

23 May 2014

Name: **Solutions**
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This is a closed-book, closed-notes quiz. Calculators are not allowed.

1. (5 point). Use the Gram-Schmidt process to generate an orthogonal set from the given linearly independent vectors:

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ -5 \\ 5 \end{bmatrix}.$$

Solution:

Let $S = w_1, w_2, w_3$ given above. Next Let $B = u_1, u_2, \ldots, u_p$ be a set of vectors in R^n. Each pair of distinct vectors of S must be orthogonal for the set S to be orthogonal. a.k.a. $(u_i)^T u_j = 0$ for all $i \neq j$.

Next define u_1, u_2, u_3 as follows:

$$u_1 = w_1$$

$$u_2 = w_2 + au_1$$

$$u_3 = w_3 + bu_1 + cu_2$$

So we have

$$u_1 = w_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Now we proceed as follows:

Multiply u_2 by $(u_1)^T$ to get

$$u_1^T u_2 = u_1^T(w_2 + au_1)$$

$$u_1^T u_2 = \begin{bmatrix}0,1,2\end{bmatrix} \begin{bmatrix}3\\6\\2\end{bmatrix} + a\begin{bmatrix}0,1,2\end{bmatrix} \begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$u_1^T u_2 = (0*3+1*6+2*2) + a(0*0+1*1+2*2)$$

$$u_1^T u_2 = 10 + 5a$$

Setting this equal to zero we get:

$$u_1^T u_2 = 10 + 5a = 0$$

$$a = -2$$

Thus

$$u_2 = \begin{bmatrix}3\\6\\2\end{bmatrix} + (-2)\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}3\\4\\-2\end{bmatrix}$$

Now we solve for u_3 by first multiplying by $(u_1)^T$ to get:

$$u_1^T u_3 = u_1^T(w_3 + bu_1 + cu_2)$$

$$u_1^T u_3 = \begin{bmatrix}0,1,2\end{bmatrix} \begin{bmatrix}10\\-5\\-5\end{bmatrix} + b\begin{bmatrix}0,1,2\end{bmatrix} \begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$u_1^T u_3 = (0*10+1*(-5)+2*5) + b(0*0+1*1+2*2) = 5+5b = 0 \rightarrow b = -1$$

Now we continue to solve for u_3 by multiplying by $(u_2)^T$ to get:

$$u_2^T u_3 = u_2^T(w_3 + bu_1 + cu_2)$$

$$u_2^T u_3 = \begin{bmatrix}3,4,-2\end{bmatrix} \begin{bmatrix}10\\-5\\-5\end{bmatrix} + c\begin{bmatrix}3,4,-2\end{bmatrix} \begin{bmatrix}3\\4\\-2\end{bmatrix}$$

$$u_2^T u_3 = (3*10+4*(-5)+(-2)*5) + c(3*3+4*4+(-2)*(-2)) = 29c = 0 \rightarrow c = 0$$

Putting all this together we get:

$$u_3 = \begin{bmatrix}10\\-5\\-5\end{bmatrix} + (-1)\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}10\\-6\\3\end{bmatrix}$$

and the set

$$S = \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}3\\4\\-2\end{bmatrix}, \begin{bmatrix}10\\-6\\3\end{bmatrix}$$

2. (5 points). Find a least squares solution to the inconsistent equation

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -7 \\ 1 & 3 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Solution:

To solve this using least squares we use the normal equation:

$$A^T A\mathbf{x} = A^T b$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 3 \\ 4 & -7 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -7 \\ 1 & 3 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 3 \\ 4 & -7 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 11 & 23 \\ 11 & 22 & 44 \\ 23 & 44 & 90 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}$$

Upon solving this system of equations we get $x_1 = -2x_3 - 3$ and $x_2 = -x_3 + 19/11$. Thus

$$\mathbf{x} = \begin{bmatrix} -2x_3 - 3 \\ -x_3 + 19/11 \\ x_3 \end{bmatrix}$$