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Lecture 12.
9 February 2015.

Hopf fibration. View S^2 as $\mathbb{CP}^1$, which parametrizes lines through 0 in $\mathbb{C}^2$. To each $\ell \in \mathbb{CP}^1$, we have the unit circle in ℓ. This leads to a fibration

$$S^1 \longrightarrow \square \longrightarrow \underset{S^2}{\overset{\mathbb{CP}^1}{\cong}},$$

which is in fact a fiber bundle. What space goes in the box?

[redacted]

More generally, $\mathbb{CP}^n$ consists of lines through 0 in $\mathbb{C}^{n+1}$. In here, we have the $(2n-1)$-sphere S^{2n+1}, which intersects every line. The topological group S^1 acts on $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ by scalar multiplication, with quotient $\mathbb{CP}^n$. Hence, we get fiber bundles

$$S^1 \to S^{2n+1} \to \mathbb{CP}^n.$$

Cell structure on $\mathbb{CP}^n$.

$(w, \sqrt{1-|w|^2}) \in \mathbb{C}^n \times \mathbb{C}$, $|w| \le 1$, with last coordinate real, ≥ 0. Get D^{2n}_+ bounded by $S^{2n-1} \subset D^{2n}_+$. Each $v \in S^{2n+1}$ is S^1-equivalent to an element of D^{2n}_+, unique if $|w| > 0$.

Hence, $\mathbb{CP}^n$ is the quotient of D^{2n}_+ by $v \sim \lambda v$ on S^{2n-1}, which is $\mathbb{CP}^{n-1}$ and a new cell.

Hence, in the $n=1$ case,

$$S^1 \to S^3 \to S^2.$$

The map $S^3 \to S^2$ is the Hopf map. If we write $S^3 \subset \mathbb{C}^4$ as (z_0, z_1), the map $S^3 \to S^2$ is

$$(z_0, z_1) \longmapsto (z_0 : z_1) = \frac{z_0}{z_1}.$$

This is the circle bundle associated to a Hermitian line bundle.

Rem. Analogous maps for $\mathbb{RP}^n$, $\mathbb{HP}^n$, $\mathbb{OP}^n$.

Cor. $\pi_n(S^3) \cong \pi_n(S^2)$ for $n \geq 3$.

Thm. $\pi_i(S^n) = 0$ for $i < n$.

Thm. $\pi_n(S^n) \cong \mathbb{Z}$. This is the degree.

Cor. $\pi_3(S^2) \cong \mathbb{Z}$.

The Hopf invariant of the Hopf map. Consider

$$f : S^3 \to S^2$$ as above.

What is C_f? It's $\mathbb{CP}^2$, whose cohomology is

$$H^*(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}[c]/(c^3).$$

Hence, $H(f) = 1$.

Ex. S^∞ is a CW complex with 2 cells in each dimension. It is contractible! Indeed, let $f: S^n \to S^\infty$ be continuous. Since S^∞ is a CW complex and S^n is compact, f factors through $S^k \subset S^\infty$ for some k. Then, $f: S^n \to S^k \to S^{k+1}$ is nullhomotopic. Hence, $\pi_n(S^\infty) = 0$ for all n. Hence, $S^\infty \to *$ is a whe, and hence a he by Whitehead.

Ex. We get $S^1 \to S^\infty \to \mathbb{CP}^\infty$, a fiber bundle. Hence, $\Omega\mathbb{CP}^\infty \simeq S^1$, and $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$.

Definition. If A is an abelian group, $n \geq 0$ an integer, a $K(A,n)$-space is a space X with $\pi_n X \cong A$ and $\pi_i X = 0$ for $i \neq n$. These turn out to be unique up to whe.

It turns out that $K(\mathbb{Z},2)$ classifies circle bundles.