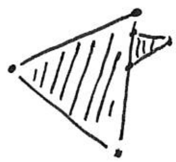
548.
Lecture 13.
11 February 2015.

<u>Def.</u> A <u>polyhedron</u> in $\mathbb{R}^n$ is a union of convex polyhedra, each a compact subset by intersecting half-planes $\sum a_i x_i \le b$.

<u>Ex.</u>



$: K$.

A <u>PL</u> map $K \longrightarrow \mathbb{R}^k$ is a map that is linear on each convex polyhedron for some decomposition. These are really <u>piecewise</u> <u>affine</u>.

<u>Ex.</u> Let $K \subseteq \mathbb{R}^n$ be a simplex, with vertices $v_0, \dots, v_n$. Specifying $f(v_0), \dots, f(v_n) \in \mathbb{R}^k$ gives $sk_0 K \longrightarrow \mathbb{R}^k$ which extends to a <u>unique</u> <u>linear</u> map $K \to \mathbb{R}^k$.

<u>Main Lemma</u>. Let $f: I^n \to Z$, $Z = W \cup_\phi D^k$ for some $\phi: S^{k-1} \to W$.

Then, f is homotopic rel $f^{-1}(W)$ to a map $f_1 : I^n \to Z$ s.t. there is a polyhedron (with nonempty interior) $K \subseteq I^n$ where

(i) $f_1(K) \subseteq e_K$ and $f_1 : K \to e_K$ is piecewise linear for some $e_K \cong \mathbb{R}^k$, and

(ii) $f_1^{-1}(U) \subset K$ for some nonempty open $U \subset e^k$.

<u>Cor</u>. $\pi_n(S^k) = 0$ for $n < k$.

<u>proof</u>. Take $W = *$, $Z = S^k$. Then, $[f] = [f_1] \in \pi_n(S^k)$. Consider $U \subseteq e^k \subseteq S^k$. By (ii), if $f_1(x) \in U$, then $x \in K$. Since $f_1|_K$ is PL, $f_1(K) \subset e_K \cong \mathbb{R}^k$ is an n-dimensional polyhedron. Hence, $f_1|_K : K \to U$ is not surjective. Hence, f_1 misses a point, and we see that $[f_1] = 0$ in $\pi_n(S^k)$.

Thm (Cellular Approximation). Every map $f: X \to Y$ between CW complexes is homotopic to a cellular map.

<u>proof of main lemma.</u> Identify $e^k \cong \mathbb{R}^k$, $B_1, B_2 \subseteq e^k$ the balls of radius 1, 2.

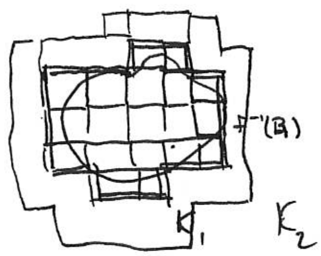
f is uniformly continuous on $f^{-1}(B_2)$. Choose $\varepsilon > 0$ so that $|x-y| < \varepsilon \Rightarrow |f(x) - f(y)| < \frac{1}{2}$.

Subdivide I so that each cube in I^n is contained in a ball of radius ε.

K_1 = union of cubes meeting $f^{-1}(B_1)$.

K_2 = union of those touching K_1.

$K_2 \subseteq f^{-1}(B_2)$, if we make ε sufficiently small.



simplexes in K_2

$g: K_2 \to e^k$ map equal to f on each vertex of subdivision.

$\phi: K_2 \to [0,1]$,

$\phi(\partial K_2) = 0$, $\phi(K_1) = 1$.

$f_t: K_2 \to e^k$

$f_t = (1 - t\phi) f + (t\phi) g$.

$f_0 = f$, $f_1|_{K_1} = g|_{K_1}$.

Extend f_t by constant homotopy on the rest of I^n.

claim: $f(\overline{I^n - K_1})$ = compact set disjoint from 0, hence from some nbd U of 0.

This proves lemma.

proof of claim:

- On $I^n - K_2$, $f_1 = f$, f takes $I^n - K_2$ outside B_1.

- For each simplex $\sigma \in K_2 - K_1$, $f(\sigma) \subseteq B_\sigma$ of radius $\frac{1}{2}$.

  Also get $g(\sigma) \subset B_\sigma$ by convexity of B_σ, and hence $f_t(\sigma) \subset B_\sigma$ for all t. Hence, $f_1(\sigma) \subset B_\sigma$. $B_\sigma \not\subset B_1$ since $\sigma \notin K_1$.

  Using radii, get that $0 \notin B_\sigma$.