Def. X has the (weak) homotopy type of a CW complex if it is (weak) homotopy equivalent to a CW complex.

Def. hCW, $wTop$, wCW. We also have $hTop$, and the pointed versions.

Remark. If X, Y are homotopy equivalent to CW complexes, the Whitehead theorem holds for maps $X \to Y$.

Thm (CW approximation). $hCW \simeq wCW \simeq wTop$.

proof. Basically, we just need to prove that every space X in Top is weak homotopy equivalent to a CW complex.

Def. (X,x) is n-connected if $\pi_i(X,x) = 0$ for $i \leq n$.

A pair (X,A) is n-connected for $n > 0$ if $\pi_i(X,A) = 0$ for $1 \leq i \leq n$ and every $(D^0, \partial D^0) \to (X,A)$ is homotopic to a map $D^0 \to A$.

Ex. S^n is $(n-1)$-connected.

Ex. (X^n, X^{n-1}) is $(n-1)$-connected. Use $\pi_{n-1}(S^n) = 0$. Prove using cellular approximation for $(D^i, \partial D^i) \to (X^n, X^{n-1})$.

Def. (X,A) a pair w/ A nonempty CW. An n-connected model for (X,A) is an n-connected pair (Z,A) with a map $Z \to X$ inducing $\mathrm{id}|_A$ s.t. $\pi_i(Z) \xrightarrow{\cong} \pi_i(X)$ for $i > n$ and an injection for $i = n$, all choices of basepoint.

Note: $\pi_i(A) \to \pi_i(Z)$ iso for $i < n$, surjection $i = n$.

$$\underset{\text{surjection}}{\pi_n(A) \to \pi_n(Z)} \underset{\text{injection}}{\to \pi_n(X)}$$

We want $n=0$, when A meets all components of X.

__Proposition__. n-connected approximations exist.

Set $Z^n = A$. Then $(A,A) \to (X,A)$ is our base case.

Assume

$$Z^n \subset Z^{n+1} \subset \cdots \subset Z^k$$

obtained by attaching cells with $(Z^k, A) \to (X,A)$ s.t.

$$\pi_i(Z^k) \to \pi_i(X) \quad \text{injection } n \le i < k,$$

$$\pi_i(Z^k) \to \pi_i(X) \quad \text{surjection } n \le i \le k,$$

all choices of basepoint.

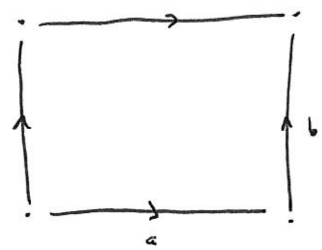
__Kill the kernel__. Attach discs D^{k+1} to kill generators of $\ker(\pi_k(Z^k) \to \pi_k(X))$, getting Y^{k+1}

__Hit the target__. Attach S^{k+1}, at basepoints to get $\pi_{k+1}(Z^{k+1}) \to \pi_{k+1}(X)$ surjection.

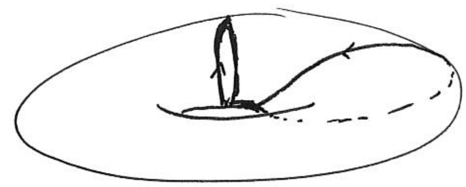
Let $Z = \bigcup_{k \ge n} Z^k$, a CW complex.

Note that $\pi_i(A) \xrightarrow{\cong} \pi_i(Z)$ for $i < n$, $\pi_n(A) \xrightarrow{\cong} \pi_n(Z)$ a surjection.

Q. What can we say about ΣM^g? What does its cell structure look like?



$aba^{-1}b^{-1}$



Attaching via commutator in $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$.

When we suspend, we attach via commutator in $\pi_2(S^2 \vee S^2)$ which is abelian!

So, $\Sigma M^1 \simeq S^2 \vee S^2 \vee S^3$.

Next up: $\pi_n(S^n) \cong \mathbb{Z}$!