<u>Thm.</u> $[X, K(A,n)]_* \cong H^n(X, A)$.

<u>sketch.</u> $$H^n(K(A,n), A) \simeq \mathrm{Hom}(H_n(A), A)$$
$$H_n(A) = \pi_n K(A,n).$$

Set $\alpha \in H^n(K(A,n), A)$ the inverse of Hurewicz.
Given $X \xrightarrow{f} K(A,n)$, get $f^*(\alpha) \in H^n(X, A)$.

Idea: it's true for $X = S^k$. So, it should be true in general.

<u>Orientability.</u> ~~[illegible]~~

$$BSO_n \to BO_n \to B\mathbb{Z}/2 \simeq K(\mathbb{Z}/2, 1)$$

fiber of $BSO_n \to (BSO_n)_1 \simeq B\mathbb{Z}/2$. This was in lecture 17.

The first Stiefel-Whitney class: $w_1(\xi) \in H^1(X, \mathbb{Z}/2)$

The obstruction to orientability.

What we'll actually prove is that the system of spaces $\{K(A,n)\}_{n\geq 0}$ forms an Ω-spectrum and hence represents a reduced cohomology theory, and we'll show that any cohomology theory h^* satisfying with

$$h^i(*) = \begin{cases} A & i=0 \\ 0 & i\neq 0 \end{cases}$$

agrees with singular cohomology with coefficients in A.

Cohomology Theories. A reduced cohomology theory for pointed CW complexes consists of a system of functors

$$\tilde{h}^n : CW_*^{op} \longrightarrow Ab$$

together with

$$\delta : \tilde{h}^n(A) \longrightarrow \tilde{h}^{n+1}(X/A)$$

for CW pairs (X,A) such that

(1) if $f \simeq g : X \to Y$, then $f^* = g^* : \tilde{h}^n(Y) \to \tilde{h}(X)$;

(2) there are natural LES:

$$\cdots \xrightarrow{\delta} \tilde{h}^n(X/A) \to \tilde{h}^n(X) \to \tilde{h}^n(A) \xrightarrow{\delta} \tilde{h}^{n+1}(X/A) \to \cdots$$

for CW pairs (X,A);

(3) the natural maps

$$\tilde{h}^n\left(\bigvee_\alpha X_\alpha\right) \longrightarrow \prod_\alpha \tilde{h}^n(X_\alpha)$$

are all isos.

Review of reduced versus non-reduced.

To get a non-reduced theory (on non-pointed CW complexes), set

$$h^n(X) = \tilde{h}^n(X_+).$$

For CW pairs, set

$$h^n(X,A) = \tilde{h}^n(X/A).$$

Given a non-reduced theory $h^n(X)$, set

$$\tilde{h}^n(X) = \operatorname{coker}\left(h^n(pt) \to h^n(X)\right)$$

induced from unique map $X \to pt$.

If X is nonempty,

$$h^n(X) \cong \tilde{h}^n(X) \oplus \underbrace{h^n(pt)}_{\tilde{h}^n(S^0)}.$$

The groups $h^n(pt) \cong \tilde{h}^n(S^0)$ are called the coefficient groups of the cohomology theory.

If $h^n(pt) \cong \tilde{h}^n(S^0) = 0$ for $n \neq 0$, $\tilde{h}$ satisfies the dimension axiom.

Q. How do you get natural maps

$$\tilde{h}^n(X) \cong \tilde{h}^{n+1}(\Sigma X)?$$

See end of Hatcher Sec. 2.3.

Use $SX \simeq \Sigma X$ for X CW, and decompose SX into two cones. Use Mayer-Vietoris, which is derived from axioms (2) and (3), and (1) since CX is contractible.

A natural question is when is a cohomology theory representable? I.e., when do we have natural isos

$$[X, K_n]_+ \cong h^n(X)?$$

One obvious desirable is

$$\begin{array}{ccc}
[X, K_n]_+ & \cong & h^n(X) \\
\wr\vert & & \wr\Vert \\
[\Sigma X, K_{n+1}] & \cong & h^{n+1}(\Sigma X) \\
\wr\Vert & & \\
[X, \Omega K_{n+1}]_+ . & &
\end{array}$$

This should be induced from $K_n \to \Omega K_{n+1}$, or

$\Sigma K_n \to K_{n+1}$.

<u>Def.</u> A <u>spectrum</u> consists of pointed spaces $\{K_n\}_{n \geq 0}$ together with maps $s_n : \Sigma K_n \to K_{n+1}$ for all $n \geq 0$.

A spectrum is an <u>Ω-spectrum</u> if the adjoint maps

$$t_n : K_n \to \Omega K_{n+1}$$

are whe.

<u>Rem.</u> In order for a spectrum to represent a cohomology theory, it must be an Ω-spectrum.