

Spectral sequences?

Definition. If $K = \{K_n\}$ is a spectrum, then the homotopy groups of K are

$$\pi_i K = \operatorname{colim}_n \pi_{i+n} K_n,$$

where $\pi_{i+n} K_n \rightarrow \pi_{i+n+1} K_{n+1}$ is induced by $K_n \rightarrow \Omega K_{n+1}$, which leads to

$$S^{i+n} \longrightarrow K_n \longrightarrow \Omega K_{n+1}$$

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$$\Sigma S^{i+n} \longrightarrow K_{n+1}$$

Σ^n

$$S^{i+n}.$$

Remark. Even if $K_n = pt$ for $n < 0$, then one can have $\pi_i K_n \neq 0$ for $i < 0$.

Exs. (1) Fix A . Set $K_n = K(A, n)$ for $n \geq 0$, $K_n = pt$ for $n < 0$. Then spectrum is the Eilenberg-MacLane spectrum HA .

$$\pi_i HA \cong \begin{cases} A & i=0, \\ 0 & i \neq 0. \end{cases}$$

Ω -spectrum.

(2) Fix a pointed space X . Set $K_n = \Sigma^n X$ for $n \geq 0$, pt for $n < 0$.

This is the suspension spectrum $\Sigma^\infty X$.

$$\pi_i \Sigma^\infty X \cong \pi_i^s X.$$

} not Ω -spectrum.

(3) For $X = S^0$, one gets $\Sigma^\infty S^0 = \mathbb{S}$ (think $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{H}_p$), the sphere spectrum, and

$$\pi_i \mathbb{S} \cong \pi_i^s.$$

Ques. Why are Ω -spectra nice? Th m.p

$$\pi_1 K_0 \longrightarrow \pi_1 K$$

is an iso. for $K = \{K_n\}_{n \in \mathbb{Z}}$ an Ω -spectrum.

Ques. Why are spectra nice? We've inverted the suspension
m.p. on spaces. Suspension becomes equivalent to shifting. Define

$$(\Sigma K)_n = \Sigma K_n,$$

with the usual bonding mps. We can also construct the loop spectrum

$$(\Omega K)_n = \Omega K_n.$$

For any space, the adjunction leads to a unit m.p

$$X \rightarrow \Omega \Sigma X.$$

It turns out that this is a stable weak equivalence, and that
at the level of spectra,

$$K \simeq \Omega \Sigma K.$$

Theorem. If K is an Ω -spectrum,

$$\tilde{h}^n(X) := [X, K_n]_+$$

is a reduced cohomology theory on ^{pointed} CW complexes.

Proof. It's trivial to check (1) and (2). Let (X, A) be a Δ pair.

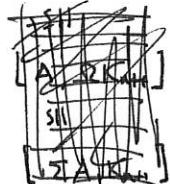
$$\begin{array}{ccccccc} A & \hookrightarrow & X & \hookrightarrow & X \cup CA & \hookleftarrow & ((X \cup CA) \cup CX) \cup C(X \cup CA) \\ \parallel & & \parallel & & \simeq j & & \simeq j \uparrow \\ A & \hookrightarrow & X & \longrightarrow & X/A & \longrightarrow & SA \longrightarrow SX \end{array} \dots$$

Cofibration sequence

$$A \hookrightarrow X \longrightarrow X/A \longrightarrow \Sigma A \longrightarrow \Sigma X \longrightarrow \Sigma(X/A) \longrightarrow \dots$$

which is natural up to homotopy. Get exact sequence

$$\dots \rightarrow [A, K_n]_+ \leftarrow [X, K_n]_+ \rightarrow [X/A, K_n]_+ \leftarrow [A, K_{n-1}]_+ \leftarrow [X, K_{n-1}]_+ \rightarrow \dots$$



$$\begin{array}{ccc} [A, \Omega K_n]_+ & & [X, \Omega K_n]_+ \\ \simeq j & & \simeq j \\ [\Sigma A, K_n]_+ & & [\Sigma X, K_n]_+ \end{array}$$