<u>Vector bundles</u>. A rank n complex vector bundle over X is an X-space $E \xrightarrow{p} X$ equipped with maps of X-spaces

i) $E \times_X E \xrightarrow{a} E$,

ii) $\mathbb{C} \times E \xrightarrow{s} E$

So, E is a locally trivial vector space in X-spaces.

such that there is an open cover $\{U_i\}_{i \in I}$ of X where $\phi_i : E_i := p^{-1}(U_i) \cong U_i \times \mathbb{C}^n$ as U_i-spaces and where i and ii are equal to vector addition and scalar multiplication:

$$E_i \times_{U_i} E_i \underset{\phi_i \times \phi_i}{\cong} U_i \times \mathbb{C}^n \times \mathbb{C}^n \xrightarrow{a} U_i \times \mathbb{C}^n,$$

$$\mathbb{C} \times E_i \underset{\sigma \times \phi_i}{\cong} \mathbb{C} \times U_i \times \mathbb{C}^n \xrightarrow{s} U_i \times \mathbb{C}^n.$$

One sees that on $U_i \cap U_j$ one has $\phi_i \circ \phi_j^{-1} : U_{ij} \times \mathbb{C}^n \to U_{ij} \times \mathbb{C}^n$ which is of the form

$$(u, v) \longmapsto (u, T_u(v))$$

for some $T_u \in GL_n(\mathbb{C})$. The assignment $u \mapsto T_u$ is continuous.

Set $\sigma_{ij} = \phi_i \circ \phi_j^{-1}$. Note that $\sigma_{ij} \cdot \sigma_{jk} = \sigma_{ik}$ on U_{ijk}. Hence, this defines a continuous Čech 1-cocycle with coefficients in $GL_n(\mathbb{C})$.

Conversely, given $\sigma_{ij} : U_{ij} \to GL_n(\mathbb{C})$ continuous maps satisfying this condition, one can <u>glue</u> to obtain a rank n complex vector bundle on X.

Exs. (1) Trivial vector bundles: $X \times \mathbb{C}^n$. A central problem is deciding whether a given vector bundle is trivial.

(2) If M is a C^∞-manifold, TM is its tangent bundle. Let $U_i \subset M$, $\underline{x} : U_i \xrightarrow{\cong} \mathbb{R}^n$ be a local coordinate system. Then, $\phi_i : TM|_{U_i} = TU_i \xrightarrow{\cong} U_i \times \mathbb{R}^n$ given by $\left(\underline{x}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$.

This is awkward. A better way to say this is that $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, with the second set of coordinates given by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. This pulls back via $\underline{x}$. Check that if $\{U_i\}_{i \in I}$ is a C^∞-chart, then one gets a cocycle.

Let $\underline{y} : U_j \xrightarrow{\cong} \mathbb{R}^n$ be compatible with $\underline{x}$. So, $\underline{x} \circ \underline{y}^{-1}$ is C^∞ on an open in $\mathbb{R}^n$ to $\mathbb{R}^n$. The induced map on tangent bundles is the matrix family $\left(\frac{\partial x_i}{\partial y_j}\right)$. The cocycle condition is expressed by the chain rule.

$$TS^n \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$$
$$\| $$
$$\{ (s,v) \mid s \in S^n \text{ and } s \cdot v = 0 \}$$

(3) If M is a C^∞-manifold, $X \subset M$ embedded there is the normal bundle $N_{X/M} = i^*TM/TX$.

$$N_{S^n/\mathbb{R}^{n+1}} \cong S^n \times \mathbb{R}.$$

Lemma. $E \to X$ is the trivial rank n vector bundle iff there are n linearly independent sections.

For the tangent bundle these are vector fields. $\Gamma(X,E)$ = vector span of vector fields.

Ex. TS^1 is trivial. TS^2 is not. Hairy ball theorem.