Non-example. Gluing along a non-linear transformation.
Get an affine bundle.

Definitions.

Given $X \xrightarrow{f} Y$, with $E \xrightarrow{p} Y$, there is a pullback $f^*E \to X$
by letting the fiber at $x \in X$ be $E_{f(x)}$.

A map of vector bundles is given by $\binom{E}{\downarrow \atop X} \to \binom{F}{\downarrow \atop Y}$ s.t.
$E \to f^*F$ is linear.

Definition. Stiefel variety $V_k(\mathbb{R}^n) \subseteq (S^{n-1})^k$ of orthonormal k-frames in $\mathbb{R}^n$.
Grassmannian $G_k(\mathbb{R}^n)$ or $G_k(\mathbb{C}^n)$ of dimension k subspaces.

What are the topologies?

$$V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n) \qquad (\text{fiber?})$$

Remark. These are manifolds and each actually an algebraic variety.

$$G_k(\mathbb{R}^n) \to G_k(\mathbb{R}^{n+1}) \to \cdots \qquad G_k(\mathbb{R}^\infty),$$

$$G_k(\mathbb{C}^n) \to G_k(\mathbb{C}^{n+1}) \to \cdots \qquad G_k(\mathbb{C}^\infty).$$

Proposition. There are universal vector bundles on these spaces.

Q. What is $G_1(\mathbb{R}^n)$, $G_1(\mathbb{C}^n)$? What is $V_1(\mathbb{R}^n)$?

Also, $V_k(\mathbb{C}^n)$ = orthonormal k-tuples for standard Hermitian metric on $\mathbb{C}^n$. $V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$.

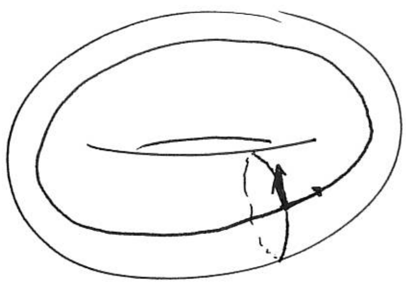
Categories. Vect - vector bundles and vector bundle morphisms.

Vect_X - vector bundles over a fixed space X. Kernel.

Whitney sum.

Q. What do maps $X \times \mathbb{C}^n \to X \times \mathbb{C}^n$ look like in Vect_X?

Ex. TX, X an elliptic curve. Oops. X a genus 1 Riemann surface.



$\cong S^1 \times S^1$

parallelizable because S^1 is.

Goal. Classification (real / complex / quaternionic) of vector bundles on all CW complexes of dimension at most 4.

A Gauss map.

$$S^n \subset \mathbb{R}^{n+1}$$

TS^n tangent bundle.

We can view each chart in TS^n as $(x, v) \in \mathbb{R}^{n+1}$.

Get $\quad S^n \longrightarrow Gr_n(\mathbb{R}^{n+1})$

Definition. A Gauss map is a monomorphism of vector bundles

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & \mathbb{C}^n \arrow[d] \\
X \arrow[r] & pt.
\end{tikzcd}$$

Hence, if rank $E = k$, we get a map $X \longrightarrow Gr_k(\mathbb{C}^n)$.
We'll use these a lot.

Theorem. Let $E \xrightarrow{f} F$ be a map of X-vector bundles. Then, f is an isomorphism (in Vect_X) if and only if it induces a v.s. iso on each fiber $x \in X$.

proof. One direction is clear.

Suppose f induces an iso on each fiber. Let g be the set-theoretic inverse. So, if

$$f_x : \overset{E_x}{\overset{\shortparallel}{\mathbb{C}^n}} \xrightarrow{\simeq} \overset{F_x}{\overset{\shortparallel}{\mathbb{C}^n}},$$

$$g_x = f_x^{-1}.$$

Q. Is g continuous? Restrict to common trivialization.

$$\begin{tikzcd} E_i \arrow[r, "\simeq", "f_i"'] \arrow[d, "\phi_i", "\wr"'] & F_i \arrow[d, "\sigma_i", "\wr"'] \\ U_i \times \mathbb{C}^n \arrow[r, dashed] & U_i \times \mathbb{C}^n \end{tikzcd}$$

$\downarrow$

continuous $\alpha_i : U_i \longrightarrow GL_n(\mathbb{C})$.

Since inverting is continuous g_i induces $\beta_i = \alpha_i^{-1}$, continuous.

So, since g is continuous locally it is continuous.