Definition. Let (X,x) be a based space. The n-fold loop space of X is

$$\Omega^n X = \mathrm{map}_*(S^n, X).$$

As in Problem 1 from HW 2, there are natural homeomorphisms

$$\Omega^n X \cong \underbrace{\Omega(\Omega(\cdots(\Omega X)))}_{n\ \Omega s}.$$

New definition. $\pi_n(X,x) \cong \pi_{n-1}(\Omega X) \cong \pi_{n-2}(\Omega^2 X) \cong \cdots \cong \pi_0(\Omega^n X)$.

Indeed,

$$[S^0, \Omega^n X]_* \cong [S^0, \Omega(\Omega(\cdots(\Omega X)\cdots))]_*$$

$$\cong [\underbrace{S^1\wedge(S^1\wedge\cdots(S^1\wedge S^1))}_{k\ S^1\text{s}}, \Omega^{n-k} X]_*$$

from k uses of adjunction

$$\cong [S^k, \Omega^{n-k} X]_*$$

$$\cong \pi_k(\Omega^{n-k} X).$$

The Eckmann–Hilton argument. We've seen that $\pi_0(X,x)$ is naturally a [redacted] set, and $\pi_1(X,x)$ is naturally a group. For the latter, we can use either that

$$\pi_1(X,x) = [S^1, X]_* \qquad \text{and } S^1 \text{ is an H-cogroup, or}$$

$$\pi_1(X,x) = [S^0, \Omega X]_* \qquad \text{and } \Omega X \text{ is an H-group.}$$

It turns out that these are the same.
(See next page.) ■

Now, suppose $n \geq 2$. Then,

$$\pi_n(X,x) \cong [S^1, \Omega^{n-1}X]_*.$$

Writing $\Omega^{n-1}X \cong \Omega(\cdots(\Omega X)\cdots)$, we see that $\pi_n(X,x)$ has two products: one from the cogroup structure on S^1, one from the group structure on $\Omega(\cdots(\Omega X)\cdots)$.

<u>Lemma</u>. Let X be a set with two binary operations $\cdot$ and $*$ such that

(i) there is a mutual two-sided identity e:

$$x \cdot e = e \cdot x = e * x = x * e = x;$$

(ii) for all x_0, x_1, y_0, y_1 in X,

$$(x_0 \cdot x_1) * (y_0 \cdot y_1) = (x_0 * y_0) \cdot (x_1 * y_1).$$

Then, $\cdot$ and $*$ are equal, associative, and commutative.

<u>Proof.</u>

$$x \cdot y = (x * e) \cdot (e * y) = (x \cdot e) * (e \cdot y) = x * y,$$

$$x \cdot (y \cdot z) = (x * e) \cdot (y * z) = (x \cdot y) * (e \cdot z) = (x \cdot y) \cdot z,$$

$$x \cdot y = (e * x) \cdot (y * e) = (e \cdot y) * (x \cdot e) = y * x = y \cdot x.$$

Lemma. The two multiplications on $\pi_1(X,*)$ agree.

proof. Write $a: [S^1,X]_* \cong [S^0,\Omega X]_*$. We need to prove that $a(f\cdot g) = a(f)*a(g)$, where

$$\underbrace{S^1 \xrightarrow{c} S^1\vee S^1 \xrightarrow{f\vee g} X\vee X \xrightarrow{id+id} X}_{f\cdot g},$$

$$\underbrace{S^0 \longrightarrow S^0\times S^0 \xrightarrow{a(f)\times a(g)} \Omega X\times \Omega X \xrightarrow{\mu} \Omega X}_{a(f)*a(g)}.$$

We just check

$$\big(a(f\cdot g)(x)\big)(t) = (f\cdot g)[t,x] = \begin{cases} f([2t,x]) & t\in[0,\tfrac12],\\ g([2t-1,x]) & t\in[\tfrac12,1],\end{cases}$$

while

$$\Big(\big(a(f)*a(g)\big)(x)\Big)(t) = \begin{cases} (a(f)(x))(t) = f([2t,x]) & t\in[0,\tfrac12],\\ (a(g)(x))(2t-1) = g([2t-1,x]) & t\in[\tfrac12,1].\end{cases}$$

<u>Proposition</u>. If G is an H-group and C is an H-cogroup, then the two multiplications on $[C, G]_*$ agree and are commutative.

<u>Cor</u>. $\pi_n(X, *)$ is abelian for $n \geq 2$.

<u>proof of proposition</u>. The unit map for both is the map from C to the basepoint of G. Thus, (i) from the lemma is satisfied.

Let $x_0, x_1, y_0, y_1 \in [C, G]_*$. Then, the diagram

$$\begin{tikzcd}
C \times C \arrow[r, "c \vee c"] \arrow[d, dash] & (C \vee C) \times (C \vee C) \arrow[r, "(x_0 \vee x_1) \times (y_0 \vee y_1)"] \arrow[d, dash] & (G \vee G) \times (G \vee G) \arrow[d, dash] \arrow[dr, "c\Delta \times c\Delta"] & \\
\Delta(C) \arrow[r] \arrow[d, dash] & (C \times *) \times (C \times *) \cup (* \times C) \times (* \times C) \arrow[r] \arrow[dd, "id_C \times \phi \times id_C"] & (G \times *) \times (G \times *) \cup (* \times G) \times (* \times G) \arrow[r] \arrow[dd, "id_G \times \phi \times id_G"] & G \times G \arrow[dr, "m"] & \\
C \arrow[d, dash, "c"'] & & & & G \\
C \vee C \arrow[r, "\Delta \vee \Delta"] & (C \times C) \vee (C \times C) \arrow[r, "(x_0 \times y_0) \vee (x_1 \times y_1)"] & (G \times G) \vee (G \times G) \arrow[r, "m \vee m"] & G \vee G \arrow[ur, dash, "c\Delta"'] &
\end{tikzcd}$$

is commutative. This proves the proposition.