Homotopy pushouts and pullbacks.

Definition. Let $\mathcal{C}$ be a category, $\{X_i\}_{i\in I}$ a collection of objects of $\mathcal{C}$, and $f_i : X \to X_i$ a collection of maps in $\mathcal{C}$. Then, X is the product of the X_i if

$$\mathrm{Hom}_{\mathcal{C}}(Y, X) \longrightarrow \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(Y, X_i) \longleftarrow \text{usual cartesian product of sets}$$

is a bijection of sets for all $Y \in \mathcal{C}$. We write $X = \prod_{i\in I} X_i$.

Dually, $X_i \xrightarrow{g_i} Y$ define a coproduct if

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \longrightarrow \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Z)$$

is a bijection for all Z. We write $Y = \coprod_{i\in I} X_i$.

Ex. In based spaces, the product is given by the usual product of spaces with the topology generated by the basis with objects

$$\prod_{i\in I} U_i$$

where $U_i \subseteq X_i$ is open and all but finitely many U_i are X_i.

The coproduct in based spaces is $\bigvee_{i\in I} X_i$, the topology around the basepoint being given by $\bigvee_{i\in I} U_i$, $x \in U_i \subseteq X_i$ open. So, for example, to give maps $X \to Z$ and $Y \to Z$ is "the same" as giving a map

$$X \vee Y \to Z.$$

__Definition__. The __pushout__ of $f: X \to Y$ and $g: X \to Z$ in $\mathcal{C}$ is an object W with morphisms $Y \xrightarrow[c_Y]{} W$ and $Z \xrightarrow[c_Z]{} W$ s.t.

(1)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "c_Y"] \\ Z \arrow[r, "c_Z"'] & W \end{tikzcd}$$
commutes;

(2) for every $d_Z: Z \to A$ and $d_Y: Y \to A$ s.t. $c_Y \circ f = c_Z \circ g$, there exists a unique $h: W \to A$ making

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "c_Y"'] \arrow[ddr, bend left, "d_Y"] & \\ Z \arrow[r, "c_Z"] \arrow[drr, bend right, "d_Z"'] & W \arrow[dr, dashed, "\exists! h"] & \\ & & A \end{tikzcd}$$

commute.

In this case, we write $Y \cup_X Z$ for W.

__Pullbacks__ are the opposite notion, written $Y \times_X Z$.

__Remark__. These are special instances of limits and colimits. They need not exist. For example, if $\mathcal{C} = \mathrm{Ab}^{\mathrm{fin}}$, the category of finite abelian groups, then

$$\bigoplus_{p \text{ prime}} \mathbb{Z}/p$$

doesn't exist. Pullbacks of manifolds don't usually exist. Let $f: M^k \to \mathbb{R}^n$ be differentiable, $x \in \mathbb{R}^n$ a non-regular point. Then,

$$\begin{tikzcd} D \arrow[r] \arrow[d] & M^k \arrow[d] \\ * \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

does not exist.

__Ex__. Villarceau circles on a torus.

Big problem. While products and coproducts are homotopy invariant, pushouts and pullbacks are not. Consider

$$\begin{tikzcd} D^n \arrow[d, "\simeq"] & S^{n-1} \arrow[l] \arrow[r] \arrow[d, "\simeq"] & D^n \arrow[d, "\simeq"] \\ * & S^{n-1} \arrow[l] \arrow[r] & * \end{tikzcd}$$

The pushouts are $S^n \longrightarrow *$, which is definitely not a homotopy equivalence.

Let X be a based space. Then, the pointed path space of X is

$$P_*X = \mathrm{map}_*\big((I^1, 0), (X, x) \big).$$

These are paths in X starting at x.

__Lemma__. P_*X is contractible.

__proof__. Define $I^1 \times P_*X \xrightarrow{h} P_*X$ by

$$h(t, \phi)(s) = \phi(ts).$$

This is a homotopy from the constant path to id_{P_*X}.

Now, consider the pullback diagrams

$$\begin{tikzcd} * \arrow[r] \arrow[d, "\simeq"] & X \arrow[d, "\simeq"] & * \arrow[l] \arrow[d, "\simeq"] \\ * \arrow[r] & X & P_*X \arrow[l, "ev_1"'] \end{tikzcd}$$

The induced map on pullbacks is $* \longrightarrow \Omega X$, which is not generally a homotopy equivalence. For example, we'll see that $\Omega S^1 \simeq \mathbb{Z}$.

The pushout
$$\begin{tikzcd} D^n \arrow[dr] & S^{n-1} \arrow[l] \arrow[r] & D^n \arrow[dl] \\ & S^n & \end{tikzcd}$$
and the pullback
$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & PX \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$
are the correct ones from the topologist's point of view. They are examples of homotopy pushouts and homotopy pullbacks.