<u>Proposition.</u> Let $p: X \to Y$ be a fibration. If $b, c \in Y$ are in the same path-component, then X_b is homotopy equivalent to X_c.

<u>proof.</u> Let $\phi: I' \to Y$ be a path. It suffices to show that $p^{-1}(\phi(0)) \simeq p^{-1}(\phi(1))$. Consider the lifting problem

$$\begin{tikzcd}
X_{\phi(0)} \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
X_{\phi(0)} \times I' \arrow[r, "h"] \arrow[ur, dashed, "g"] & Y
\end{tikzcd},$$

where $h(x,t) = \phi(t)$. Since p is a fibration, a lift g exists, and defines a map $X_{\phi(0)} \to X_{\phi(1)}$ by restriction to $X_{\phi(0)} \times \{1\}$ and commutativity.

Suppose $\phi \simeq \gamma$ rel $\partial I'$. We get two maps $X_{\phi(0)} \to X_{\phi(1)}$. It turns out that they're homotopic. We pick

$$\begin{tikzcd}
X_{\phi(0)} \times I \times \{0\} \cup X_{\phi(0)} \times \partial I' \times I' \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
X_{\phi(0)} \times I' \times I' \arrow[r, "k"'] \arrow[ur, dashed, "m"] & Y
\end{tikzcd},$$

inclusion of $X_{\phi(0)}$ on $X_{\phi(0)} \times I' \times \{0\}$, and the two lifts as above for ϕ, γ on $X_{\phi(0)} \times \partial I' \times I'$.

where $k(x,s,t) = j(s,t)$, a homotopy from ϕ to γ rel $\partial I'$.

Note that $(I \times I, I \times \{0\} \cup \partial I \times I) \cong (I \times I, I \times \{0\})$.

Hence fibrations also satisfy this lifting property, and a map m as in the diagram exists. $m(x,1,t)$ is the desired homotopy. In particular, $\phi \circ \phi^{-1} \simeq \mathrm{id}_{X_{\phi(1)}}$ and $\phi^{-1} \circ \phi \simeq \mathrm{id}_{X_{\phi(0)}}$.

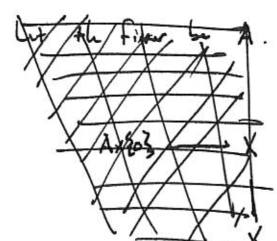
If Y is a space, $\mathrm{Top}_{/Y}$ is the category of spaces with a fixed map to Y, as $f: X \to Y$. In $\mathrm{Top}_{/Y}$, the notion of homotopy is fiber-wise. For example, two spaces $f: X \to Y$ and $g: Z \to Y$ over Y are <u>homotopy equivalent over Y</u>, or fiber-wise homotopy equivalent, if there are maps $a: X \to Y$ and $b: Y \to Z$ in $\mathrm{Top}_{/Y}$ s.t. $a \circ b \simeq id_Y$ and $b \circ a \simeq id_Z$ <u>in</u> $\mathrm{Top}_{/Y}$. So, one has various commutation diagrams

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[dr, "f"'] & Z \arrow[d, "g"] \\ & Y \end{tikzcd}, \qquad \begin{tikzcd} Z \arrow[r, "b"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}, \qquad \begin{tikzcd} & X \times \partial I' \arrow[dl] \arrow[dr, "id_X \sqcup b\circ a"] & \\ X \times I' \arrow[rr] \arrow[dr] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}.$$

<u>Proposition</u>. If Y is locally contractible, then any fibration $X \xrightarrow{p} Y$ is locally fiber homotopy equivalent to a product fibration.

<u>proof</u>. We can assume Y is contractible, so that $id_Y \simeq * \in Y$. It suffices then to show that if $A \times I' \xrightarrow{h} Y$, then $h_0^* p \simeq h_1^* p$ over A. The idea is to follow the argument in the previous proposition:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "\phi(0)\text{ fiber}"] \arrow[d] & h^* X \arrow[d] \\ A \times I' \arrow[r, "(id, \phi)"'] \arrow[ur, dashed] & A \times I' \end{tikzcd}.$$



This automatically preserves fibers.

<u>Motto</u>. Fibrations are homotopy fiber bundles.

<u>Proposition</u>. Let $f : X \to Y$ be a pointed map, $P_* f$ the homotopy fiber, pointed by $(*, c_*)$, the constant loop at the basepoint. For any pointed space Z, the sequence

$$[Z, P_* f]_* \xrightarrow{i_*} [Z, X]_* \xrightarrow{f_*} [Z, Y]_*$$

is exact, where i_* is induced from $P_* f \underset{j}{\subseteq} Pf \to X$.

<u>proof</u>. There is a commutative diagram

$$\begin{tikzcd}
 & {[Z, X]_*} \arrow[r] & {[Z, Y]_*} \\
{[Z, P_* f]_*} \arrow[r] & {[Z, Pf]_*} \arrow[u, "\cong"] \arrow[r, "ev"] & {[Z, Y]_*,} \arrow[u, equal]
\end{tikzcd}$$

from which we see that we can just work with the bottom row and show exactness there.

Clearly if $[g] \in [Z, Pf]_*$ is $j_*[f]$, then $ev_*[g] = [*]$. Suppose that $ev_*[g] = [*]$. Since $Pf \to Y$ is a fibration, there exists a lift

$$\begin{tikzcd}
Z \times \{0\} \arrow[r, "g"] \arrow[d] & Pf \arrow[d] \\
Z \times I' \arrow[r, "h"'] \arrow[ur, "k"] & Y,
\end{tikzcd}$$

where h is a homotopy from g to $*$. Hence, k is a homotopy from g to a map $Z \to P_* f \subseteq Pf$.

<u>Remark</u>. If $Z = S^n, n \geq 1$, these are exact sequences of groups.