MATH 215 – Sets (S)

Definition 1 *A* **set** *is a collection of objects. The objects in a set X are called* **elements** *of X.*

Notation 2 *A set can be described using* **set-builder notation**. *That is, a set can be described by writing $\{x : P(x)\}$ where $P(x)$ is a statement concerning an object x. For example, $\mathbb{N} = \{x : x \text{ is a natural number}\}$ or $(2, 7] = \{x : \text{ is a real number and } 2 < x \leq 7\}$.*

Remark 3 *Some authors use the notation $\{x|P(x)\}$ in place of $\{x : P(x)\}$.*

Notation 4 *The set that contains no elements is called the* **empty set** *and is denoted $\varnothing$. [Note: $\{\varnothing\}$ is NOT the empty set.]*

Notation 5 *If an object x is an element of a set A, then we write $x \in A$. If x is not an element of A, then we write $x \notin A$.*

Definition 6 *Two sets A and B are equal if and only if every element of A belongs to B and every element of B belongs to A.*

Definition 7 *If A and B are sets such that each element of A is also an element of B, then we say A is a* **subset** *of B, and notate this $A \subseteq B$.*

Remark 8 *Some authors use the notation $A \subset B$ if A is a subset of B. We reserve the notation $\subset$ for something else later.*

Proposition 9 *Let A and B be sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.*

Definition 10 *Let A be a set. The* **power set** *of A, denoted $P(A)$ or 2^A, is the set of all subsets of A.*

Conjecture 11 *Let A be a set with n elements, where n is some natural number.*

1. *Make a conjecture as to how many elements $P(A)$ has.*
2. *Prove your conjecture using induction on n.*

Definition 12 *If A and B are sets, we say that A is a* **proper subset** *of B, written $A \subset B$ if $A \subseteq B$ and $A \neq B$.*

Problem 13 *Let A be a set with n elements, where n is some natural number. How many* proper *subsets does A have?*

Definition 14 *Let A and B be sets. The* **union** *of A and B, denoted $A \cup B$, is defined as the set of all elements in either A or B (or both). In set-builder notation*

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Definition 15 *Let A and B be sets. The* **intersection** *of A and B, denoted $A \cap B$, is defined as the set of all elements in both A and B. In set-builder notation*

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Definition 16 *Two sets A and B are* **disjoint** *if $A \cap B = \varnothing$.*

Problem 17 *Find two sets A and B where $A \cup B = \{1, 2, 3\}$ and $A \cap B = \{3\}$.*

Proposition 18 *Let A, B, C be sets. Then*

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Definition 19 *Let X and Y be sets. Define the* **difference** *$X - Y$ of X and Y by*

$$X - Y = \{x : x \in X \text{ and } x \notin Y\}.$$

Remark 20 *The union and intersection of sets can be defined for more than two sets. In fact, the union and intersection can be defined for an infinite number of sets.*

Definition 21 *Let I be an index set and let $\{X_i\}_{i \in I}$ be a collection of sets. Then the* **union** *of the collection $\{X_i\}_{i \in I}$ is defined*

$$\bigcup_{i \in I} X_i = \{x : x \in X_i \text{ for some } i \in I\}.$$

Definition 22 *Let I be an index set and let $\{X_i\}_{i \in I}$ be a collection of sets. Then the* **intersection** *of the collection $\{X_i\}_{i \in I}$ is defined*

$$\bigcap_{i \in I} X_i = \{x : x \in X_i \text{ for all } i \in I\}.$$

Example 23 *Let $\mathbb{Z}$ be the index set and for $n \in \mathbb{Z}$, define $X_n = [n, n+1)$.*

1. $\displaystyle\bigcup_{n \in \mathbb{Z}} X_n$
2. $\displaystyle\bigcap_{n \in \mathbb{Z}} X_n$

3. $X_i \cap X_j$ *for* $i \neq j$

4. *If* Y_n *is defined as* $Y_n = [n, n+1]$, *find* $Y_i \cap Y_j$ *for* $i \neq j$.

Definition 24 *Let* X *be a set. Let* I *be an index set and let* $\{X_i\}_{i \in I}$ *be a collection of subsets of* X. *We say that the subsets in the collection are* **mutually disjoint** *if for all* $i, j \in I$ *where* $i \neq j$, *we have* $X_i \cap X_j = \varnothing$. *A* **disjoint union** *is the union of subsets in a collection that are mutually disjoint, denoted by*

$$\coprod_{i \in I} X_i.$$

Definition 25 *Let* X *be a set, let* I *be an index set, and let* $\{X_i\}_{i \in I}$ *be a collection of mutually disjoint subsets of* X. *We say the collection is a* **partition** *of* X *if*

$$\coprod_{i \in I} X_i = X.$$

Example 26 *Let* $\mathbb{R}$ *the set of all real numbers. Then the collection of sets* $X_n = [n, n+1)$ *for* $n \in \mathbb{Z}$ *is a partition of* $\mathbb{R}$. *Note that the collection of sets* $Y_n = [n, n+1]$ *for* $n \geq 1$ *is NOT a partition of* $\mathbb{R}$. *Why?*

Proposition 27 *Every set* S *has a partition.*

Problem 28 *Find a partition of* $\mathbb{Z}$, *the set of integers.*

Proposition 29 *Let* n *be a natural number. Let* $A_0, A_1, A_2, \ldots, A_{n-1}$ *be defined by*

$$A_i = \{x \in \mathbb{Z} : x \equiv i \bmod n\}, \ \ i = 0, 1, 2, \ldots, n-1.$$

Then $A_0, A_1, \ldots, A_{n-1}$ *is a partition of* $\mathbb{Z}$.

We will return to partitions after we introduce more terminology.

Definition 30 *Let* X *and* Y *be sets. Let* $x \in X$ *and* $y \in Y$. *Define the* **ordered pair** (x, y) *of* x *and* y *by*

$$(x, y) = \{\{x\}, \{x, y\}\}.$$

Proposition 31 *Let* X *and* Y *be sets. Let* $x, x' \in X$ *and* $y, y' \in Y$. *Then* $(x, y) = (x', y')$ *if and only if* $x = x'$ *and* $y = y'$.

Remark 32 *This proposition seems obvious to us, however now we have a rigorous definition of what an ordered pair is (i.e. in terms of sets) that we must use to prove it.*

Definition 33 *Let* X *and* Y *be sets. The* **Cartesian product** *of* X *and* Y, *denoted* $X \times Y$, *is the set of all ordered pairs* (x, y) *such that* $x \in X$ *and* $y \in Y$. *In set notation,*

$$X \times Y = \{(x, y) : x \in X, \ y \in Y\}.$$

Example 34 *Let $X = Y = \mathbb{R}$. Then*

$$\mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\} = \mathbb{R}^2.$$

Problem 35 *Let $\mathbb{R}_+$ denote the set of positive real numbers. What is the difference between $\mathbb{R} \times \mathbb{R}_+$ and $\mathbb{R}_+ \times \mathbb{R}$ in terms of $\mathbb{R}^2$?*

Definition 36 *Let X and Y be sets. A* **relation** *between X and Y is a subset of $X \times Y$. If we denote a relation between X and Y by $R(X, Y)$, then we have*

$$R(X, Y) \subseteq X \times Y.$$

Example 37 *Here are some examples of relations.*

1. *Let $X = Y = \mathbb{R}$ and $R(X, Y) = \{(x, y) \in X \times Y : y = x^2\}$.*

2. *Let $X = \mathbb{Z} - \{0\}$ and $Y = \mathbb{N}$. Let $R(X, Y) = \{(x, y) \in X \times Y : y = x^2\}$.*

3. *Let $X = Y = \mathbb{Z}$, and let n be a natural number. Define a relation $R(X, Y) = R(X, X) = \{(x, y) : x \equiv y \bmod n\}$. A relation $R(X, X)$ between X and itself is called a* **relation on** *X, and sometimes we denote the relation by $x \sim_R y$ (or simply $x \sim y$). In this example, we have that $x \sim_R y$ if and only if $x \equiv y \bmod n$.*

4. *Let P denote the set of points in the Euclidean plane. Let $\sim$ be the relation on P defined by $p \sim q$ if and only if p and q are a pair of points in the plane that are the same distance from the origin.*

5. *Let P be as in the previous example, and let $X = P - \{(0, 0)\}$. Let $\sim$ be the relation on X defined by $p \sim q$ if and only if p and q lie on the same line through the origin.*

6. *Let $X = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Define a relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad - bc = 0$.*

7. *$X = Y = \mathbb{R}$. Define $x \sim y$ if and only if $x < y$.*

There are two fundamental types of relations we will be interested in: **equivalence relations** and **maps (or functions)**. Both types of relations satisfy extra properties beyond simply being a relation.

Definition 38 *Let X be a set and $\sim$ a relation on X.*

- *The relation $\sim$ is called* **reflexive** *if $x \sim x$ for all $x \in X$.*

- *The relation $\sim$ is called* **symmetric** *if $x \sim y$ implies $y \sim x$ for all $x, y \in X$.*

- *The relation $\sim$ is called* **transitive** *if $x \sim y$ and $y \sim z$ implies $x \sim z$ for all $x, y, z \in X$.*

If a relation is reflexive, symmetric, and transitive, it is called an **equivalence relation**.

Problem 39 *Determine which of the relations in Example 37 are equivalence relations.*

Definition 40 *Let X be a set and let $\sim$ be an equivalence relation on X. For each $x \in X$, the* **equivalence class of** *x, denoted $[x]$, is the subset of X defined by*

$$[x] = \{y \in X : x \sim y\}.$$

If $[x]$ is an equivalence class in X and $z \in [x]$, we say that z is a **representative** *of $[x]$. Note that an equivalence class can have more than one representative.*

Problem 41 *Determine the equivalence classes of the equivalence relations in Example 37.*

Theorem 42 *Let X be a set.*

1. *If $\sim$ is an equivalence relation on X, then the set of distinct equivalence classes defined by $\sim$ form a partition of X.*

2. *Conversely, if $\{X_i\}_{i \in I}$ is a partition of X into non-empty, mutually disjoint subsets of X, then the relation on X defined by $x \sim y$ if and only if $x, y \in X_i$ for some i is an equivalence relation.*

Definition 43 *Let X and Y be sets. A* **map** *f from X to Y, denoted*

$$f : X \to Y$$

is a relation between X and Y with the property that for every $x \in X$, there exists a unique $y \in Y$ such that (x, y) is in the relation.

Notation 44

1. *We will write $f(x)$ for y when referring to a map f.*
2. *If (x, y) is an element of the relation defined by f then we write*

$$x \mapsto f(x).$$

3. *What is wrong with writing the following? What should it be?*

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \sqrt{x} \end{aligned}$$

Problem 45 *Determine which relations in Example 37 are maps.*

Definition 46 *Let X and Y be sets and let $f : X \to Y$ be a map.*

1. *The set X is called the* **domain** *of f.*
2. *The set Y is called the* **co-domain***, or* **target***, of f.*
3. *Given a subset $W \subseteq X$ of the domain, we call*

$$f(W) = \{y \in Y : \text{there exists } x \in W \text{ such that } y = f(x)\}$$

the **image of W under f***. Here are two important special cases.*

   (a) *When $W = \{x\}$ consist of a single element in X, we call $f(\{x\})$ the* **image of x under f***. We then abuse notation and write $f(x)$ in place of $f(\{x\})$.*

   (b) *When $W = X$, we call*

$$f(X) = Im\ f$$

the **image of f** *or the* **range of f***.*

4. *Given a subset $Z \subseteq Y$ of the target, we call*

$$f^{-1}(Z) = \{x \in X : f(x) \in Z\}$$

the **preimage of Z under f***. Here are two important special cases.*

   (a) *When $Z = \{y\}$ consist of a single element, we call*

$$f^{-1}(\{y\}) = \{x \in X : f(x) = y\}$$

the **preimage of y under f***. We then abuse notation and write $f^{-1}(y)$ in place of $f^{-1}(\{y\})$.*

(b) *When $Z = Y$, then $f^{-1}(Y) = X$.*

Example 47 *Determine which of the following examples are maps and which are not. If it is not a map, explain why. Then, if it is a map, state the domain and range of the map (where possible).*

1. *Let n be a natural number and let $\mathbb{Z}/n\mathbb{Z}$ denote the n distinct equivalence classes of congruence modulo n. Define*

$$\begin{aligned} f : \mathbb{Z} &\to \mathbb{Z}/n\mathbb{Z} \\ a &\mapsto [a], \end{aligned}$$

where $[a]$ is the equivalence class containing a.

2. *Define*

$$\begin{aligned} f : \mathbb{Z}/n\mathbb{Z} &\to \mathbb{Z} \\ [a] &\mapsto a. \end{aligned}$$

3. *Let $\mathbb{R}_+ = [0, \infty)$ denote the set of non-negative real numbers. Define*

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R}_+ \\ (x, y) &\mapsto x^2 + y^2. \end{aligned}$$

4. *Let X be any set and define the* **identity map on** *X, Id_X, by*

$$\begin{aligned} Id_X : X &\to X \\ x &\mapsto x. \end{aligned}$$

5. *Let f, g be maps from a set X to X. Define the* **composition of** *f* **with** *g as*

$$\begin{aligned} f \circ g : X &\to X \\ x &\mapsto f(g(x)). \end{aligned}$$

Proposition 48 *Let $f : X \to Y$ be a map. The following are equivalent:*

1. *For all $y \in Y$, the preimage $f^{-1}(y)$ consists at most one element.*

2. *For all $x, x' \in X$, if $f(x) = f(x')$, then $x = x'$.*

3. *For all $x, x' \in X$, if $x \neq x'$, then $f(x) \neq f(x')$.*

Definition 49 *If a map satisfies any of the equivalent statements in Proposition 48, then the maps is called* **injective** *or* **one-to-one**.

Example 50 *Which of the maps in Example 47 are injective?*

Example 51 *Which of the following are injections?*

1. $f : \mathbb{Z} \to \mathbb{Z}$ *defined by* $f(x) = x^2$.
2. $f : \mathbb{Z} \to \mathbb{Z}$ *defined by* $f(x) = x^3$.
3. $f : [0, \infty) \to \mathbb{R}$ *defined by* $f(x) = x^2$.
4. $f : \mathbb{R} \to \mathbb{R}^3$ *defined by* $f(t) = (\cos t, \sin t, t)$.
5. $\sin : \mathbb{R} \to \mathbb{R}$.
6. $\sin : [0, \pi/2] \to [-1, 1]$.
7. $\sin : [-\pi/2, \pi/2] \to [-1, 1]$.

Definition 52 *A function* $f : \mathbb{R} \to \mathbb{R}$ *is* **strictly increasing** *(***strictly decreasing***) if for all* $x_1, x_2 \in \mathbb{R}$, $f(x_1) < f(x_2)$ $(f(x_1) > f(x_2))$.

Problem 53 *True or False: A strictly increasing or strictly decreasing function is injective.*

Proposition 54 *Let* $f : X \to Y$ *be a map. The following are equivalent:*

1. *For all* $y \in Y$, *the preimage* $f^{-1}(y)$ *consists at least one element.*
2. *For all* $y \in Y$, *there exists* $x \in X$ *such that* $y = f(x)$.
3. *Im* $f = Y$.

Definition 55 *If a map satisfies any of the equivalent statements in Proposition 54, then the maps is called* **surjective** *or* **onto**.

Problem 56 *Determine which of the maps in Example 47 and Example 51.*

Example 57 *Let* X *and* Y *be sets. Define the* **projection onto** X *by*

$$\begin{aligned} \pi_X : X \times Y &\to X \\ (x, y) &\mapsto x \end{aligned}$$

and **projection onto** Y *by*

$$\begin{aligned} \pi_Y : X \times Y &\to Y \\ (x, y) &\mapsto y. \end{aligned}$$

Show that both π_X *and* π_Y *are surjective functions.*

Definition 58 *A map that is both an injection and a surjection is called a* **bijection**.