MATH 215 – Practice Final

Definition 1 *Let X be a set. Let I be an index set and let $\{X_i\}_{i\in I}$ be a collection of subsets of X. We say that the subsets in the collection are* **mutually disjoint** *if for all $i, j \in I$ where $i \neq j$, we have $X_i \cap X_j = \varnothing$. A* **disjoint union** *is the union of subsets in a collection that are mutually disjoint, denoted by*

$$\coprod_{i\in I} X_i.$$

Definition 2 *Let X be a set, let I be an index set, and let $\{X_i\}_{i\in I}$ be a collection of mutually disjoint subsets of X. We say the collection is a* **partition** *of X if*

$$\coprod_{i\in I} X_i = X.$$

Definition 3 *Let X and Y be sets. The* **Cartesian product** *of X and Y, denoted $X \times Y$, is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$. In set notation,*

$$X \times Y = \{(x, y) : x \in X,\ y \in Y\}.$$

Example 4 *Let $X = Y = \mathbb{R}$. Then*

$$\mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\} = \mathbb{R}^2.$$

Definition 5 *Let X and Y be sets. A* **relation** *between X and Y is a subset of $X \times Y$. If we denote a relation between X and Y by $R(X, Y)$, then we have*

$$R(X, Y) \subseteq X \times Y.$$

Example 6 *Let n be a natural number. Define a relation $R(\mathbb{Z}, \mathbb{Z}) = \{(x, y) : x \equiv y \bmod n\}$.*

A relation $R(X, X)$ between X and itself is called a **relation on** X, and sometimes we denote the relation by $x \sim_R y$ (or simply $x \sim y$). In this example, we have that $x \sim_R y$ if and only if $x \equiv y \bmod n$.

Definition 7 *Let X be a set and $\sim$ a relation on X.*

- *The relation $\sim$ is called* **reflexive** *if $x \sim x$ for all $x \in X$.*
- *The relation $\sim$ is called* **symmetric** *if $x \sim y$ implies $y \sim x$ for all $x, y \in X$.*
- *The relation $\sim$ is called* **transitive** *if $x \sim y$ and $y \sim z$ implies $x \sim z$ for all $x, y, z \in X$.*

If a relation is reflexive, symmetric, and transitive, it is called an **equivalence relation**.

Problem 8 *For each of the following determine whether the relation is reflexive, symmetric, and transitive.*

1. *Let* $X = Y = \mathbb{R}$ *and* $R(X, Y) = \{(x, y) \in X \times Y : y = x^2\}$.

2. *Let* $X = \mathbb{Z} - \{0\}$ *and* $Y = \mathbb{N}$. *Let* $R(X, Y) = \{(x, y) \in X \times Y : y = x^2\}$.

3. *Let P denote the set of points in the Euclidean plane. Let $\sim$ be the relation on P defined by $p \sim q$ if and only if p and q are a pair of points in the plane that are the same distance from the origin.*

4. *Let P be as in the previous example, and let $X = P - \{(0, 0)\}$. Let $\sim$ be the relation on X defined by $p \sim q$ if and only if p and q lie on the same line through the origin.*

5. *Let $X = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Define a relation $\sim$ on X by $(a, b) \sim (c, d)$ if and only if $ad - bc = 0$.*

6. $X = Y = \mathbb{R}$. *Define $x \sim y$ if and only if $x < y$.*

Proposition 9 *The relation $R(\mathbb{Z}, \mathbb{Z}) = \{(x, y) : x \equiv y \bmod n\}$ from Example 6 is an equivalence relation on $\mathbb{Z}$.*

Definition 10 *Let X be a set and let $\sim$ be an equivalence relation on X. For each $x \in X$, the* **equivalence class of** *x, denoted $[x]$, is the subset of X defined by*

$$[x] = \{y \in X : x \sim y\}.$$

If $[x]$ is an equivalence class in X and $z \in [x]$, we say that z is a **representative** *of $[x]$. Note that an equivalence class can have more than one representative.*

Problem 11 *Determine the equivalence classes of the equivalence relations in Example 6. How many equivalence classes are there?*

Theorem 12 *Let X be a set.*

1. *If $\sim$ is an equivalence relation on X, then the set of distinct equivalence classes defined by $\sim$ form a partition of X.*

2. *Conversely, if $\{X_i\}_{i \in I}$ is a partition of X into non-empty, mutually disjoint subsets of X, then the relation on X defined by $x \sim y$ if and only if $x, y \in X_i$ for some i is an equivalence relation.*