Rational Homotopy Theory - Lecture 2

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This lecture contains brief mentions of four additional applications of rational homotopy as well as a brief recollection of homotopy theory.

1. Geodesics

Conjecture 1.1. *Let M be a closed Riemannian manifold of dimension at least 2. Then, there are infinitely many closed geodesics on M.*

Here a geodesic is closed if it is closed as a 1-manifold, so it's a loop in M. Two geodesics are considered distinct if they trace out different subsets of M.

Note that it is a non-trivial theorem, due to Lyusternik and Fet from 1951 that any closed Riemannian manifold M has at least 1 closed geodesic. When $\pi_1(M) \neq 0$, any non-zero homotopy class $0 \neq [\gamma] \in \pi_1(M)$ is represented by a closed geodesic. This is proved by minimizing the length functional $\ell : \Lambda M \to \mathbb{R}$, where ΛM is the space of smooth maps $S^1 \to M$. The functional is given by

$$\ell(\gamma) = \int_{S^1} \langle \gamma', \gamma' \rangle \, \mathrm{ds},$$

where γ is a differentiable loop in M and the brackets are those of the Riemannian metric. This functional turns out to behave like a Morse function, even though ΛM is an infinite-dimensional manifold. Nevertheless, its properties are good enough for the following theorem.

Theorem 1.2 (Gromoll-Meyer [6])**.** *If M is a 1-connected closed Riemannian manifold having only finitely many closed geodesics, then the Betti numbers $\beta_i(\Lambda M) = \dim_{\mathbb{Q}} \mathrm{H}^i(\Lambda M, \mathbb{Q})$ are bounded: there exists an integer N such that $\beta_i(\Lambda M) \leq N$ for all i.*

Corollary 1.3. *If the Betti numbers of ΛM are not bounded, then M has infinitely many closed geodesics.*

In many cases it is possible to check directly using spectral sequences that the Betti numbers of ΛM are unbounded. The next theorem does better and uses the computational power of rational homotopy theory, as we will see later in the course.

Theorem 1.4 (Sullivan, Vigué-Poirrier [7])**.** *If M is a 1-connected CW complex (such as a 1-connected closed manifold), and if $\mathrm{H}^*(M, \mathbb{Q})$ requires at least 2 generators as a $\mathbb{Q}$-algebra, then the Betti numbers $\beta_i(\Lambda M)$ are unbounded.*

Corollary 1.5. *If M is a losed Riemannian manifold with $\pi_1(M)$ finite and such that $\mathrm{H}^*(M, \mathbb{Q})$ requires at least 2 generators as a $\mathbb{Q}$-algebra, then there are infinitely many closed geodesics on M.*

Remarkably, the hypothesis on the $\mathbb{Q}$-cohomology of M does not depend on the chosen Riemannian metric on M! The main cases where the corollary does not apply are for spheres S^n and complex projective spaces $\mathbb{CP}^n$. At present, the existence of infinitely many closed geodesics on these spaces is unknown. It is a result of Bangert [2] from 1993 that there are infinitely many closed geodesics on any Riemannian structure on S^2. Interestingly, it had been known for many years that any such 2-sphere had at least 3 closed geodesics, which had been a conjecture of Poincaré from 1905.

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2. Local complete intersections

There are two results of interest for the moment. Recall that a map $R \to S$ of commutative rings is **smoothable** if it can be factored as $R \to Q \to S$ where Q is smooth over R and $Q \to S$ is surjective. At the level of affine schemes, this says that $\operatorname{Spec} S$ can be realized as a closed subscheme of the smooth R-scheme $\operatorname{Spec} Q$. A smoothable morphism is a **local complete intersection morphism** (**lci** for short) if for some (and it turns out any) smoothing $R \to Q \xrightarrow{h} S$ as above, the following condition is satisfied: for each prime ideal $\mathfrak{p}$ of S, the kernel of $Q_{\mathfrak{p}'} \to P_{\mathfrak{p}}$ is generated by a $\mathfrak{p}'$-regular sequence, where $\mathfrak{p}' = h^{-1}(\mathfrak{p})$.

Recall that in a commutative ring Q, given a prime $\mathfrak{q}$, a sequence of elements $f_1, \ldots, f_c \in \mathfrak{q}$ is $\mathfrak{q}$-regular if for $2 \leq i \leq c$, the image of f_i in $Q/(f_1, \ldots, f_{i-1})$ is not a zero-divisor. Hence, the standard example of a local complete intersection over a field k, for example, looks locally like $k[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ where the f_i form a regular sequence.

Local complete intersection morphisms have many nice properties as outlined by the two theorems below, both proved using methods in rational homotopy theory.

Theorem 2.1 (Avramov-Halperin [1]). *If $f : R \to S$ is a smoothable morphism of commutative $\mathbb{Q}$-algebras such that S has finite Tor-dimension over R, then the cotangent complex L_f is of bounded Tor-amplitude if and only if f is a lci morphism.*

The proof goes by using rational homotopy theory to study the 'fiber' of f. The next theorem is about the growth of resolutions. Let R be a noetherian commutative local k-algebra for some field k containing $\mathbb{Q}$ such that $R/\mathfrak{m} \cong k$.

Theorem 2.2 (Félix-Thomas [5]). *Suppose that R is a Noetherian graded-commutative k-algebra where k is a field containing $\mathbb{Q}$, and suppose that $R_0 = k$. Then, the radius of convergence of*

$$P_R(z) = \sum_{i=0}^{\infty} \dim_k \operatorname{Tor}_i^R(k, k) z^i$$

is

(1) $+\infty$ *if and only if R is a polynomial algebra,*
(2) 1 *if and only if R is a lci,*
(3) < 1 *if and only if R is not a lci.*

3. The elliptic-hyperbolic dichotomy

Definition 3.1. A simply connected n-dimensional finite CW complex is **elliptic** if $\pi_k(X)_{\mathbb{Q}} = 0$ for $k \geq 2n$. A finite CW complex is **hyperbolic** if the ranks of the vector spaces $\pi_k(X)_{\mathbb{Q}}$ grow exponentially.

Theorem 3.2. *A simply connected finite CW complex is either elliptic or hyperbolic.*

A reference for this theorem is the book of Félix-Halperin-Thomas [4].

4. A brief recollection of homotopy theory

Two maps $f, g : X \to Y$ are **homotopic** (or sometimes **homotopy equivalent**) if there is another map $h : X \times I^1 \to Y$ such that $h|_{X \times \{0\}} = f$ and $h|_{X \times \{1\}} = g$. In this case, h is a **homotopy** from f to g, and we write $f \simeq g$ if f and g are homotopic. It is easy to check that homotopy is an equivalence relation on the set of maps from X to Y, and we write $[X, Y]$ for the set of homotopy classes of maps from X to Y.

Two spaces X and Y are **homotopy equivalent** if there exist maps $f : X \rightleftarrows Y : g$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. In this case we write simply $X \simeq Y$. A space X is **contractible** if $X \simeq *$.

There are pointed versions of these notions, where each $h_t = h|_{X \times \{t\}}$ is required to be a pointed map. The set of homotopy classes of pointed maps from (X, x) to (Y, y) will be denoted by $[(X, x), (Y, y)]_*$, or $[X, Y]_*$ when the basepoints are implicit.

If (X, x) is a pointed space, the homotopy groups are $\pi_n(X, x) = [(S^n, s), (X, x)]_*$, where for concreteness we let $S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} | \sum x_i^2 = 1\}$, and $s = (1, 0, \dots, 0)$. Here are some basic facts about these objects.

(1) The set $\pi_0(X, x)$ is viewd as a pointed set. It is the pointed set of path-components of X, pointed by the component containing x.
(2) The set $\pi_n(X, x)$ is naturally a group for $n \geq 1$, and it is abelian if $n \geq 2$.
(3) The set $\pi_n(X, x)$ does not depend on x if X is path-connected.
(4) The homotopy groups are functorial in pointed spaces and take pointed homotopy equivalences to isomorphisms.

Example 4.1. (1) $\pi_k(S^n) = 0$ for $0 \leq k < n$.
(2) $\pi_n(S^n) \cong \mathbb{Z}$ for $n \geq 1$.
(3) The groups $\pi_k(S^n)$ when $k > n$ are notoriously difficult to compute.

A map $f : X \to Y$ is a **weak homotopy equivalence** if for all $x \in X$ the induced map $\pi_n(f) : \pi_n(X, x) \to \pi_n(Y, f(x))$ is a bijection for all $n \geq 0$.

Exercise 4.2. Show that every homotopy equivalence is a weak homotopy equivalence.

It is not the case that the converse is true. For example, there exist non-contractible spaces whose homotopy groups vanish identically. These are pathologies and won't be important in this course.

The topological spaces that interest us the most are those that are homotopy equivalent to CW complexes, and hence can be thought of as being built out simple cells. This process is akin to choosing a projective resolution of a module and makes computations much easier.

Let A be a topological space, and let $f : S^{n-1} \to A$ be a map. The pushout $X = A \cup_f D^n$, or $X = A \cup_{S^{n-2}} D^n$, is said to be obtained by attaching a **cell** $(D^n \to X)$ to A via the **attaching map** f. Note that the pushout is obtained as the quotient of $A \coprod D^n$ by the equivalence relation $x \sim f(x)$ when $x \in S^{n-1} \subseteq D^n$ and $f(x) \in A$.

Example 4.3. If $A = *$ and $f : S^{n-1} \to *$, the pushout is S^n. A different approach is to start with S^{n-1} and attach two cells D^n_+ and D^n_- to S^{n-1}, both via the identity map $S^{n-1} \to S^{n-1}$. The result is S^n where each new cell is a hemisphere.

Exercise 4.4. Find an attaching map $S^3 \to S^2$ such that the pushout is homeomorphic to $\mathbb{CP}^2$.

The next definition is one of the most important.

Definition 4.5. A CW structure on a space X consists of a sequence $X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X$ of closed subspaces such that

(i) X^0 is a set of closed points of X,
(ii) X^n is obtained from X^{n-1} by attaching cells D^n_α along a set of attaching maps $\{\phi_\alpha : S^{n-1} \to X^{n-1}\}_{\alpha \in A}$, and
(iii) the induced map $\cup_n X^n \to X$ is a homeomorphism.

In particular, the topology on X is the topology where a set $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all $n \geq 0$. The subspace X^n is called the n-skeleton of X, although it should be noted that this is not a homotopy invariant notion, as the example above shows. Rather, it depends on the choice of cell structure.

In practice, all spaces of interest that seem to come up either themselves admit CW structures or are homotopy equivalent (not just weakly homotopy equivalent) to spaces that do.

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