Rational Homotopy Theory - Lecture 4

BENJAMIN ANTIEAU

1. Coconnected commutative differential graded algebras

Let k be a field of characteristic 0. We will be interested in cohomological commutative dg k-algebras A with $A^n = 0$ for $n < 0$. These are the commutative algebra objects in $\mathrm{Ch}_k^{\geq 0}$, the category of non-negatively graded cochain complexes of k-modules. Verbosely, I will call these **strictly coconnective** k-cdgas. A k-cdga A is called **coconnective** if $\mathrm{H}^n(A) = 0$ for $n < 0$.

If moreover the unit map $k \to \mathrm{H}^0(A)$ is an isomorphism, then we say that A is **coconnected**, and if $A^n = 0$ for $n < 0$ *and* $k \xrightarrow{\cong} A^0$, then A is **strictly coconnected**.

Question 1.1. Is every coconnective cdga A quasi-isomorphic to a strictly coconnective cdga B? Is every coconnected cdga A quasi-isomorphic to a strictly coconnected cdga B?

I don't know the answer to this question at the moment. But, some care is definitely needed. Consider the good trunaction $\tau^{\geq 0} A$ which has

$$(\tau^{\geq 0} A)^n = \begin{cases} 0 & \text{if } n < 0, \\ A^0/\mathrm{im}(A^{-1} \to A^0) & \text{if } n = 0, \text{ and} \\ A^n & \text{if } n > 0. \end{cases}$$

The good truncation has the property that the natural map $\mathrm{H}^n(A) \to \mathrm{H}^n(\tau^{\geq 0} A)$ is an isomorphisms for $n \geq 0$, while $\mathrm{H}^n(\tau^{\geq 0} A) = 0$ for $n < 0$. However, and this is the key point, it is not in general possible to put an algebra structure on $\tau^{\geq 0} A$ when A is a cdga such that $A \to \tau^{\geq 0} A$ is an algebra map.

Example 1.2. Consider $A = k[u, u^{-1}]$ where $|u| = 2$ with zero differential. The map $\mathrm{H}^*(A) \to \mathrm{H}^*(\tau^{\geq 0} A)$ would have to kill a unit, so the target would have to be the zero cdga.

This issue won't be important for us, as we'll always be able model our homotopy types by strictly coconnective cdgas.

Example 1.3. If M is a differentiable manifold, then $\mathrm{A}^\bullet_{\mathrm{dR}}(M)$ is coconnected if and only if M is connected. The de Rham complex is strictly coconnected (as an $\mathbb{R}$-cdga) if and only if M is a point.

We will in fact only need to use strictly coconnective $\mathbb{Q}$-cdgas in the rational homotopy theory we develop.

Example 1.4. Let $\Lambda_n(x_1, \ldots, x_r)$ be the graded polynomial algebra over k on classes $x_1, \ldots, x_r$ of degree n if n is even, and let it be the exterior algebra on classes x_i of degree n if n is odd.

Convention 1.5. If I write $\Lambda_n(x_1, \ldots, x_r)$ without specifying a differential, then implicitely I mean that $d = 0$. Other times, I will say for example let A be $\Lambda_1(x, y, z)$ with $d(x) = d(y) = 0$ and $d(z) = xy$ (the Heisenberg cdga).

As above, let k be a field of characteristic zero. We will abuse notation and write Ch_k for the category of cochain complexes over k. The forgetful functor $\mathrm{Ch}_k \leftarrow \mathrm{CDGA}_k : U$ has a left adjoint, which we will write as $\mathrm{Sym}_k(M)$ when M is a chain complex. The existence of

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$\mathrm{Sym}_k : \mathrm{Ch}_k \to \mathrm{CDGA}_k$ follows formally from category theory, specifically from the adjoint functor theorem.

The category Ch_k of cochain complexes has an invertible endofunctor $\Sigma : \mathrm{Ch}_k \to \mathrm{Ch}_k$ called **suspension**. The cochain complex ΣM is given by $(\Sigma M)^n = M^{n+1}$, while the differential $d_{\Sigma M} = -d_M$. More generally, $(\Sigma^t M)^n = M^{n-t}$ and $d_{\Sigma^t M} = (-1)^t d_M$.

Exercise 1.6. Let V be a k-module with basis $\{x_1, \ldots, x_r\}$ show that $\Lambda_n(y_1, \ldots, y_s) \cong \mathrm{Sym}_k(\Sigma^{-n} V)$ for $n \in \mathbb{Z}$.

We will also write $\Lambda_n(V)$ for $\mathrm{Sym}_k(\Sigma^{-n} V)$.

2. The canonical filtration

Let A be a strictly coconnective k-cdga. Let $A(m)$ denote the sub-cdga generated by x and $d(x)$ for $x \in A^i$ where $0 \leq i \leq m$. Let $A(-1) = k \subseteq A^0$. Clearly, the filtration $A(-1) \subseteq A(0) \subseteq A(1) \subseteq \cdots$ is exhaustive in that every $x \in A$ in $A(m)$ for some m.

Exercise 2.1. Check that the inclusion $A(m) \to A$ is indeed a map of cdgas.

Lemma 2.2. *The natural map $A(m) \to A$ induces an isomorphism $\mathrm{H}^i(A(m)) \to \mathrm{H}^i(A)$ for $0 \leq i \leq m$ and an injection $\mathrm{H}^{m+1}(A(m)) \to \mathrm{H}^{m+1}(A)$.*

Proof. Since $A(m)^i = A^i$ for $0 \leq i \leq m$, the first claim holds for $0 \leq i < m$. In degree m, we just have to observe that if $x \in A(m)^m = A^m$, then $d(x) = 0$ in $A(m)$ if and only if $d(x) = 0$ in A^m by the construction of $A(m)$. We also see that the groups $\mathrm{B}^{m+1}(A(m))$ and $\mathrm{B}^{m+1}(A)$ of boundaries in degree $m = 1$ are naturally isomorphic. Moreover, $\mathrm{Z}^{m+1}(A(m)) \subseteq \mathrm{Z}^{m+1}(A)$ since $A(m) \to A$ is a map of cdgas. ☐

Let $A(m, q)$ for $m \geq 0$ and $q \geq 0$ be defined inductively by $A(m, 0) = A(m-1)$ and by letting $A(m, q)$ be the subalgebra of A generated by $A(m, q-1)$ and

$$\{x \in A^m : d(x) \in A(m, q-1)\}$$

for $q \geq 1$. Note that $A(m, q)$ is a sub-cdga of A.

Example 2.3. Let $A = \Lambda_1(x, y)$, where $d(y) = xy$. Then, $A(1) = A$, but $A(1, q) = \Lambda_1(x)$ for all $q \geq 1$. This shows that the filtration $A(m, q)$ need not be exhaustive.

In the previous example, we see that the natural map $A' = \Lambda_1(x) \to A$ is a quasi-isomorphism and the canonical filtrations on A' are exhaustive. It turns out that this exhaustivity property is extremely important for coconnective cdgas.

Definition 2.4. a strictly connective k-cdga A is **minimal** if

1. (1) k is strictly connected,
2. (2) the underlying graded-commutative algebra A is free (i.e., a graded-polynomial ring), and
3. (3) $\cup_{q \geq 0} A(n, q) = A(n)$ (the filtrations are exhaustive).

Thus, we see that Example 2.3 is not minimal: it satisfies (1) and (2) but not (3). Of course, the quasi-isomorphism $\Lambda_1(x) \to A$ in that example is exactly expressing the fact that we have not chosen a minimal set of generators needed to produce the cohomology of A.

Example 2.5. The Heisenberg cdga $A = \Lambda_1(x, y, z)$ with $d(x) = d(y) = 0$ and $d(z) = xy$ is minimal. We see that $A(1) = A$, so we have only to check condition (3) for $n = 1$. However, $A(1, 1) = \Lambda_1(x, y)$, and $d(z) \in A(1, 1)$, so $A(1, 2) = A$.

Example 2.6. If A is strictly connected and $d_A = 0$, then A is minimal if and only if it is (graded) free.

The minimal k-cdgas play a central role in what is to come.

Exercise 2.7. Determine whether or not the following algebras are minimal:

- (i) $\Lambda_1(x) \otimes_k P_2(y, z)$ where $d(x) = 0$, $d(z) = 0$, and $d(y) = xz$;

(ii) $\Lambda_1(x) \otimes_k P_2(y, z)$ where $d(x) = 0$, $d(z) = 0$, and $d(y) = xy$;
(iii) $\Lambda_1(x) \otimes_k P_2(y, z)$ where $d(x) = z$, $d(z) = 0$, and $d(y) = xz$;
(iv) $\Lambda_3(x) \otimes_k P_2(y, z)$ where $d(x) = z^2$, $d(z) = 0$, and $d(y) = x$;
(v) $\Lambda_3(x) \otimes_k P_2(y, z)$ where $d(x) = z^2$, $d(z) = 0$, and $d(y) = 0$.

As these exercises show, minimal algebras are cdgas that can be built up step by step by adding new generators in increasing order by degree.

Exercise 2.8. Find minimal $\mathbb{Q}$-cdgas with the following cohomology algebras:

(i) $\mathrm{H}^*(S^n, \mathbb{Q})$;
(ii) $\mathrm{H}^*(\mathbb{CP}^n, \mathbb{Q})$;
(iii) $\mathrm{H}^*(T^n, \mathbb{Q})$, where $T^n = (S^1)^{\times n}$ is the n-torus;
(iv) $\mathrm{H}^*(\mathbb{CP}^m \times \mathbb{CP}^n, \mathbb{Q})$.

Definition 2.9. Let $A \hookrightarrow B$ be an inclusion of striclty connective cdgas. We say that B is an **elementary extension of** A if as graded-commutative algebras, $B = A \otimes_k \Lambda_n(V)$ for some k-module V and $d(V) \subseteq A$.

Elementary extensions are completely controlled by the connecting map $d : V \to A^{n+1}$, and we write $A \otimes_d \Lambda(V)$ for B in this case.

Example 2.10. The Heisenberg cdga is $\Lambda_1(x, y) \otimes_d \Lambda_1(z)$ with $d(z) = xy$.

Lemma 2.11. *Any minimal cdga is obtained from k by a (possibly transfinite) sequence of elementary extensions.*

Proof. Each $A(n, q)$ is an elementary extension of $A(n, q-1)$ for $q \geq 1$. Hence, the result follows from the exhaustivity of the filtrations and the fact that the graded-commutative algebra underlying a minimal algebra is free. □

Exercise 2.12. Show that if A and B are minimal k-cdgas, then $A \otimes_k B$ is a minimal k-cdga.

3. Indecomposables

Let A be a strictly coconnected k-cdga. There is a canonical augmentation $\epsilon : A \to k$ of k-cdgas. Write A^+ for the kernel of the augmentation, and let

$$\mathrm{Q}A = A^+/(A^+ \cdot A^+)$$

be the **indecomposables** in A. Since $d(A^+ \cdot A^+) \subseteq A^+$ by the Leibniz rule, the differential on A induces a differential on $\mathrm{Q}A$.

Definition 3.1. If A is a strictly coconnected k-cdga, the homotopy group $\pi^n A$ is defined to be $\mathrm{H}^n(\mathrm{Q}A)$. Evidently, $\pi^n A = 0$ for $n \leq 0$.

This definition will be amply justified later.

Example 3.2. If $A = \Lambda_1(x) \otimes_d \Lambda_1(y)$ where $d(y) = xy$, then $\mathrm{Q}^1 A$ is a 2-dimensional vector space with basis $\{x, y\}$, while $\mathrm{Q}^n A = 0$ for $n \neq 1$. In particular, $\pi^1 A = k^{\oplus 2}$, and all other homotopy groups vanish. Recall that the inclusion $\Lambda_1(x) \to A$ is a quasi-isomorphism. But, $\pi^1 \Lambda_1(x)$ is 1-dimensional.

The previous example shows that $\pi^* A$ is *not* homotopy invariant. In other words, taking homotopy groups does not take quasi-isomorphisms of strictly coconnected k-cdgas to isomorphisms of abelian groups. However, we will see that $\pi^* A$ is a homotopy invariant of *minimal* k-cdgas.

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