Rational Homotopy Theory - Lecture 6

BENJAMIN ANTIEAU

1. Minimal models

Theorem 1.1. *Let A be a coconnected k-cdga. There exists a minimal k-cdga M and a quasi-isomorphism $M \to A$.*

Proof. We let $M(0) = k$, which maps canonically to A. We will construct $M(n, q)$ inductively, for $n \geq 1$ and $q \geq 0$. We set $M(1, 0) = M(0) = k$. Suppose that we have constructed $f = f(n, q) : M(n, q) \to A$, which induces isomorphisms $\mathrm{H}^m(M(n, q)) \xrightarrow{\cong} \mathrm{H}^m(A)$ for $0 \leq m < n$ and an injection $\mathrm{H}^n(M(n, q)) \to \mathrm{H}^n(A)$. Choose cocycles $g_i \in A^n$ whose images in $\mathrm{H}^n(A)$ map to a basis for the cokernel of $\mathrm{H}^n(M(n, q)) \to \mathrm{H}^n(A)$. Let $h_j \in M(n, q)^{n+1}$ be cocycles whose images in $\mathrm{H}^{n+1}(M(n, q))$ generate the kernel of $\mathrm{H}^{n+1}(M(n, q)) \to \mathrm{H}^{n+1}(A)$. Finally, let $c_j \in A^n$ satisfy $d(c_j) = f(h_j)$.

Let $M(n, q + 1) = M(n, q) \otimes_k \Lambda_n(x_i) \otimes_d \Lambda_n(y_j)$, where $d(y_j) = h_j$. There is an obvious map $f(n, q + 1) \to M(n, q + 1)$ given by sending x_i to g_i and y_j to c_j. It is easy to see that $\mathrm{H}^m(M(n, q)) \cong \mathrm{H}^m(M, q+1)$ for $0 \leq m < n$, while $\mathrm{H}^n(M(n, q)) \to \mathrm{H}^n(M(n, q+1)) \to \mathrm{H}^n(A)$ are injections.

Let $M(n) = \cup_{q \geq 0} M(n, q)$. There is a natural map $f(n) : M(n) \to A$ from commutativity, and clearly the induced map $\mathrm{H}^m(M(n)) \to \mathrm{H}^m(A)$ is an isomorphism for $0 \leq m \leq n$ and an injection for $m = n + 1$. Specifically, we have that

$$\mathrm{H}^{n+1}(M(n)) = \operatorname*{colim}_q \mathrm{H}^{n+1}(M(n, q)) \to \mathrm{H}^{n+1}(A).$$

Any element $x \in \mathrm{H}^{n+1}(M(n))$ is in the image of $x' \in \mathrm{H}^{n+1}(M(n, q))$ for some q. If x (and hence x') maps to zero in $\mathrm{H}^{n+1}(A)$, x' maps to zero in $\mathrm{H}^{n+1}(M(n, q + 1))$.

Finally, let $M = \cup_{n \geq 0} M(n)$. The natural map $f : M \to A$ is a quasi-isomorphism by construction. □

We will call M a **minimal model** for A. That is, a minimal model for a connected k-cdga is a minimal k-cdga M together with a fixed quasi-isomorphism $f : M \to A$. The theorem proves that every connected k-cdga has a minimal model.

Remark 1.2. Note how the process would be much easier if $\mathrm{H}^1(M) = 0$.

Exercise 1.3. I don't see an easy way to prove the following, and we will prove it later using the homotopy theory of k-cdgas. However, you might want to give it a try. Let $f : M \to N$ be a quasi-isomorphism of *minimal* k-cdgas. Prove that f is an isomorphism.

Exercise 1.4. Show that a minimal model for $\mathrm{A}_{\mathrm{dR}}(S^n)$ has $d = 0$ if and only if $n \geq 1$ is odd.

To get a sense for why the proof is a little subtle, we considered the problem of running the proof to find a k-cdga for the graded-commutative ring

$$A = (\Lambda_1(x) \otimes_k \Lambda_3(y)) / (xy),$$

viewed as a cdga with $d = 0$. The first step is $M(1, 0) = k$, and the next is $M(1, 1) = \Lambda_1(s)$ with $M(1, 1) \to A$ given by sending s to x. This is already surjective on degree 1 cohomology, and it is injective on degree 2 cohomology. So, $M(1, q) = M(1, 1)$ for $q \geq 1$. Additionally, $M(2, q) = M(1, 1)$ for all $q \geq 0$, and hence $M(3, 0) = M(1, 1)$.

Date: 28 January 2016.

The next step is to adjoint a class t that will map to y. So, set $M(3,1) = \Lambda_1(s) \otimes_k \Lambda_3(t) = M(1,1) \otimes_k \Lambda_3(t)$, where $d(t) = 0$. The map $M(3,1) \to A$ sends t to y. This has the property that it is an isomorphism up to degree 3 cohomology. However, the map on degree 4 cohomology is not injective, as the class represented by st maps to zero in A.

So, we set $M(3,2) = M(3,1) \otimes_d \Lambda_3(u)$, where $d(u) = st$, and let $M(3,2) \to A$ be defined by sending u to 0. This algebra solves the problem of the non-injectivity on H^4 owing to st, but it creates a new problem: both su and tu represent cohomology classes (in H^4 and H^6, respectively) that now go to 0 in A. So, we must let $M(3,3) = M(3,2) \otimes_d \Lambda_3(v)$, where $d(v) = su$. This solves the su problem but creates an sv problem.

Exercise 1.5. Show that the $M(3,q) \neq M(3,q+1)$ for any q.

So, we see that the method of the proof is required and might involve infinitely given steps in any given degree.

Exercise 1.6. Consider the orientable manifold $S^1 \times S^3$. What is its $\mathbb{R}$-cohomology ring? Is is possible to find another manifold M with a map to $S^1 \times S^3$ such that $\mathrm{H}^*(M,\mathbb{R})$ is isomorphic to $A = (\Lambda_1(x) \otimes_k \Lambda_3(y))/(xy)$? If so, give an example. If not, say why not, and give an example of a CW complex with this property if possible.

2. Indecoposables and generators of a minimal cdga

Lemma 2.1. *Let M be a minimal k-cdga. Then, as a graded-commutative k-algebra, M is isomorphic to the free algebra on $\pi^*M \cong \mathrm{Q}^*M$.*

Proof. Pick representatives x_i in M of a homogeneous basis for Q^*M. There is an induced map from the graded-commutative algebra F generated freely by the x_i to M. It is surjective by the definition of Q^*M. However, it is injective because M is free as a graded-commutative algebra and $\mathrm{Q}^*F \to \mathrm{Q}^*M$ is an isomorphism. □

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