Rational Homotopy Theory - Lecture 9

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1. The homotopy category of a model category

Exercise 1.1. Let $f : X \to Z$ be a map between CW complexes, and take the mapping path space $X \xrightarrow{f} Pf \xrightarrow{p} Y$, where

$$Pf = \{(\lambda, x) \in Y^{I^1} \times X : \lambda(0) = f(x)\}.$$

The inclusion $X \to Pf$ as constant paths is obviously a weak homotopy equivalence, even a homotopy equivalence. Moreover, $Pf \to Y$ is a Serre fibration. Some more work is needed to replace $X \to Pf$ by a cofibration.

Now, we come to the main reason why model categories have been so successful in encoding homotopical ideas: the homotopy category of a model category.

Definition 1.2. Let M be a category and W a class of morphisms in M. The localization of M by W, if it exists, is a category $M[W^{-1}]$ with a functor $L : M \to M[W^{-1}]$ such that

(1) $L(w)$ is an isomorphism for every $w \in W$,
(2) every functor $F : M \to N$ having the property that $F(w)$ is an isomorphism for all $w \in W$ factors uniquely through L in the sense that there is a functor $G : M[W^{-1}] \to N$ and a natural isomorphism of functors $G \circ L \simeq F$, and
(3) for any category N, the functor $\mathrm{Fun}(M[W^{-1}], N) \to \mathrm{Fun}(M, N)$ induced by composition with $L : M \to M[W^{-1}]$ is fully faithful.

The localization of M by W, if it exists, is unique up to categorical equivalence.

In general, there is no reason that a localization of M by W should exist much less be useful. The fundamental problem is that in attempting to concretely construct the morphisms in $M[W^{-1}]$, for example by hammock localization (hat piling), one discovers size issues, where it might be necessary to enlarge the universe in order to obtain a category: the morphisms sets in a category must be actual sets, not proper classes.

Theorem 1.3 ([4])**.** *Let M be a model category with class of weak equivalences W. Then, the localization $M[W^{-1}]$ exists. It is called the homotopy category of M, and we will denote it by* $\mathrm{Ho}(M)$.

We will not prove this theorem in full, but rather we will give a detailed sketch with parts to be filled in as exercises. We follow [1] very closely.

Definition 1.4. Let M be a model category, and fix $X \in M$. A **cylinder object** for X is an object $I \wedge X$ together with maps

$$X \coprod X \to I \wedge X \xrightarrow{\simeq} X$$

such that the composition $X \coprod X \to X$ is the **folding map**. The cylinder object is **good** if $X \coprod X \to I \wedge X$ is a cofibration, and it is **very good** if additionally $I \wedge X \to X$ is a fibration (necessarily acyclic).

Lemma 1.5. *Every object X of a model category M has a very good cylinder object.*

Proof. Take a factorization $X \coprod X \to I \wedge X \to X$ as in **M4** where $X \coprod X \to I \wedge X$ is a cofibration and $I \wedge X \to X$ is a fibration. □

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Definition 1.6. Two maps $f, g : X \to Y$ are **left homotopic** if there exists a cylinder object $I \wedge X$ for X and a map $h : I \wedge X \to Y$ such that $h(i_0 \coprod i_1) = f \coprod g$. There are similar notions of two maps being **good left homotopic** and **very good left homotopic**. We will write $f \sim_l g$ when f and g are left homotopic.

Example 1.7. If $M = \text{Top}$, then $I^1 \times X$ is a cylinder object for X, so that classically homotopic maps are in particular left homotopic. When is $I^1 \times X$ good?

Warning 1.8. In general, right homotopy is *not* an equivalence relation on the set $\text{Hom}_M(X, Y)$. However, this will be the case in important special cases, as we'll see below.

Exercise 1.9. Show that if f is a weak equivalence and $g \sim_l f$, then g is a weak equivalence.

Lemma 1.10. *Any left homotopic maps $f \sim_l g : X \to Y$ are good left homotopic. If Y is fibrant, then they are very good left homotopic.*

There is a certain inertia to proofs in model category theory.

Proof. Consider the diagram $X \coprod X \to I \wedge X \xrightarrow{h} Y$, which exhibits a left homotopy between f and g. Take a factorization

$$X \coprod X \to (I \wedge X)' \xrightarrow{\simeq} I \wedge X \xrightarrow{h} Y,$$

where the first map is a cofibration. Then, $(I \wedge X)'$ is a good cylinder object for X, and the composition $(I \wedge X)' \xrightarrow{\simeq} I \wedge X \xrightarrow{h} Y$ is a good homotopy from f to g. Now, if Y is fibrant, choose a further cylinder object $(I \wedge X)' \to (I \wedge X)'' \to X$ by an $(W \cap C, F)$-factorization. Then, the homotopy $h' : (I \wedge X)' \to Y$ extends to $(I \wedge X)''$ since $Y \to *$ is a fibration. □

Lemma 1.11. *If X is cofibrant and $I \wedge X$ is a good cylinder object, then the maps $i_0, i_1 : X \to I \wedge X$ are acyclic cofibrations.*

Proof. The maps $i_0, i_1 : X \to X \coprod X$ are cofibrations as they are pushouts of cofibrations. Since compositions of cofibrations are cofibrations, this shows that $i_0, i_1 : X \to I \wedge X$ are cofibrations. Now, use the two-out-of-three property **M1**. □

Lemma 1.12. *If X is cofibrant, then $\sim_l$ is an equivalence relation on* $\text{Hom}_M(X, Y)$ *for any Y.*

Proof. Reflexivity follows from the fact that X a cylinder object for itself. Symmetry follows from the fact that the switch map on $X \coprod X$ is an isomorphism, so we can precompose a homotopy $I \times X \to Y$ with the switch map. Transitivity is the more interesting property. Take the pushout of good homotopies from f to g and from g to k. □

Whether or not X is cofibrant, $\pi^l(X, Y)$ will denote the quotient of $\text{Hom}_M(X, Y)$ be the equivalence relation *generated by* left homotopy.

Lemma 1.13. *If X is cofibrant and $p : Y \to Z$ is an acyclic fibration, then $\pi^l(X, Y) \to \pi^l(X, Z)$ is a bijection.*

Proof. The hypothesis imply that $\text{Hom}_M(X, Y) \to \text{Hom}_M(X, Z)$ is a surjection, so the same is true of $\pi^l(X, Y) \to \pi^l(X, Z)$. Suppose that two maps $f, g : X \to Y$ become left homotopic after composition with p. Pick a good homotopy from $p \circ f$ to $p \circ g$. This homotopy lifts to Y by **M3**, since $X \coprod X \to I \wedge X$ is a cofibration. □

Lemma 1.14. *Let Z be fibrant, $f \sim_l g : Y \to Z$ two left homotopic maps, and $X \xrightarrow{i} Y$ a morphism. Then, $i \circ f \sim_l i \circ g$.*

Proof. Use a very good homotopy between f and g and a good cylinder object for X. □

Exercise 1.15. Prove that composition of morphisms induces a well-defined composition $\pi^l(X, Y) \times \pi^l(Y, Z) \to \pi^l(X, Z)$ whenever Z is fibrant.

References

[1] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126.

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[5] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.