Rational Homotopy Theory - Lecture 10

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1. The homotopy category of a model category continued

See the writeup [1] from Dwyer and Spalinski, where most of the lecture was taken.

2. Some computational remarks

Recipe 2.1. It is generally difficult to compute $[X, Y] = \mathrm{Hom}_{\mathrm{Ho}(M)}(X, Y)$ given two objects $X, Y \in M$. We give a recipe. Replace X by a weakly equivalent cofibrant object QX, and Y by a weakly equivalent fibrant object RY. Then, $[X, Y] = \mathrm{Hom}_M(QX, RY)/\sim$, where $\sim$ is an equivalence relation on $\mathrm{Hom}_M(QX, RY)$ generalizing homotopy equivalence (see [2, Section II.1]). See [1, Proposition 5.11] for a proof that this construction does indeed computing the set of maps in the homotopy category.

Remark 2.2. In many cases, every object of M might be cofibrant, in which case one just needs to replace Y by QY and compute the homotopy classes of maps. This is for example the case in sSets.

Remark 2.3. In Goerss-Jardine [2, Section II.1], the homotopy category $\mathrm{Ho}(M)$ is itself defined to be the category of objects of M that are both fibrant and cofibrant, with maps given by $\mathrm{Hom}_{\mathrm{Ho}(M)}(A, B) = \mathrm{Hom}(A, B)/\sim$. Given an arbitrary X in M it is possible to assign to X a fibrant-cofibrant object RQX as follows. First, take, via **M4**, a factorization $\emptyset \to QX \to X$ where QX is cofibrant $QX \to X$ is a weak equivalence. Now, take a factorization $QX \to RQX \to *$ of the canonical map $QX \to *$ in which $QX \to RQX$ is an acyclic cofibration and $RQX \to *$ is a fibration. In particular, RQX is fibrant. Since compositions of cofibrations are cofibrations, RQX is also cofibrant. Moreover, if $f : X \to Y$ is a morphism, then it is possible using **M3** to (non-uniquely) assign to f a morphism $RQf : RQX \to RQY$ such that one gets a well-defined functor $M \to \mathrm{Ho}(M)$ (i.e., after enforcing $\sim$).

Remark 2.4. In practice, we will work with simplicial model category structures, for which there exist objects $QX \times \Delta^1$, where Δ^1 is the standard 1-simplex (so that $|\Delta^1| = I^1$). In this case, the equivalence relation $\sim$ is precisely that of (left) homotopy classes of maps.

Exercise 2.5. For chain complexes, the equivalence relation $\sim$ is precisely that of chain homotopy equivalence. (See [5, Section 1.4].) Using the recipe above, compute

$$\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Ch}_{\geq 0}(\mathbb{Z}))}(\mathbb{Z}/p[1], \mathbb{Z}),$$

where $\mathbb{Z}/p[1]$ denotes the chain complex with $\mathbb{Z}/p$ placed in degree 1 and zeros elsewhere.

References

[1] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126.

[2] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999.

[3] P. Goerss and K. Schemmerhorn, *Model categories and simplicial methods*, Interactions between homotopy theory and algebra, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, pp. 3–49.

[4] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin-New York, 1967.

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[5] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.