Rational Homotopy Theory - Lecture 13

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1. Quillen equivalences

Definition 1.1. Consider a pair of adjoint functors

$$F : M \rightleftarrows N : G$$

between model categories M and N. The pair is called a **Quillen pair**, or a a pair of Quillen functors, if one of the following equivalent conditions is satisfied:

- F preserves cofibrations and acyclic cofibrations;
- G preserves fibrations and acyclic fibrations.

In this case, F is also called a **left Quillen functor**, and G a **right Quillen functor**.

Quillen pairs provide a sufficient framework for a pair of adjoint functors on model categories to descend to a pair of adjoint functors on the homotopy categories.

Proposition 1.2. *Suppose that $F : M \rightleftarrows N : G$ is a pair of Quillen functors. Then, there are functors $\mathbf{L}F : M \to \mathrm{Ho}(N)$ and $\mathbf{R}G : N \to \mathrm{Ho}(M)$, each of which takes weak equivalences to isomorphisms, such that there is an induced adjunction $\mathbf{L}F : \mathrm{Ho}(M) \rightleftarrows \mathrm{Ho}(N) : \mathbf{R}G$ between homotopy categories.*

Proof. See [2, Theorem 9.7]. □

Remark 1.3. The familiar functors from homological algebra all arise in this way, so $\mathbf{L}F$ is called the left derived functor of F, while $\mathbf{R}G$ is the right derived functor of G. There is a recipe for computing the value of the derived functors on an arbitrary object X of M and Y of N. Specifically, $\mathbf{L}F(X)$ is isomorphic to $F(QX)$ where $QX \to X$ is an acyclic fibration with QX cofibrant in M. Similarly, $\mathbf{R}G(Y)$ is weakly equivalent to $G(RY)$ where $Y \to RY$ is an acyclic cofibration with RY fibrant in N.

Example 1.4. Consider the adjoint functors $\mathrm{Ch}_{\geq 0}(\mathbb{Z}) \rightleftarrows \mathrm{Ch}_{\geq 0}(\mathbb{Q})$ given by rationalization on the left and forgetting on the right, where these model categories are given the projective model category structure. Note that this forgetful functor trivially preserves quasi-isomorphisms and positive degree-wise surjections, so the pair is derivable by the proposition. Note that it is worth noting that these categories are locally presentable, so the existence of one adjoint is checkable by the adjoint functor theorem. For details, see [1].

Definition 1.5. A **Quillen equivalence** is a Quillen pair $F : M \rightleftarrows N : G$ such that $\mathbf{L}F : \mathrm{Ho}(M) \rightleftarrows \mathrm{Ho}(N) : \mathbf{R}G$ is an inverse equivalence.

Proposition 1.6. *In the situation of the previous proposition, if in addition for every morphism $f : A \to G(X)$ in M, where A is cofibrant and X is fibrant, the conditions that f and the adjoint $F(A) \to X$ are weak equivalences are equivalent, then $\mathbf{L}F$ and $\mathbf{R}G$ are inverse equivalences.*

Proof. Again, see [2, Theorem 9.7]. □

Theorem 1.7. *Geometric realization and the singular set functor form a Quillen equivalence pair*

$$|-| : \mathrm{sSets} \rightleftarrows \mathrm{Top} : \mathrm{Sing}\,.$$

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Proof. The pair is derivably by definition of the model categories involved. Suppose that X is a fibrant topological space, i.e., any space. Let A be a cofibrant simplicial set, i.e., any simplical set. Finally, let $f : A \to \mathrm{Sing}(X)$ be a morphism. If f is a weak equivalence, this means that $|f| : |A| \to |\mathrm{Sing}(X)|$ is a weak homotopy equivalence. However, the counit map $|\mathrm{Sing}(X)| \to X$ is clearly a weak homotopy equivalence by cellular approximation. So, it follows that the composition $|A| \to X$ is a weak homotopy equivalence, and Proposition (1.6) applies. □

2. Mapping Spaces

This section is borrowed from my forthcoming notes on motivic homotopy theory with Elden Elmanto.

We will now explain simplicial model categories since we will need to discuss mapping spaces. For details, we refer the reader to [3, II.2-3]. Let X, Y be simplicial sets, then we may define the **simplicial mapping space** $\mathrm{map}_{\mathrm{sSets}}(X, Y)$ as the simplicial set whose set of n-simplices is given by

$$\mathrm{map}_{\mathrm{sSets}}(X, Y)_n := \mathrm{Hom}_{\mathrm{sSets}}(X \times \Delta^n, Y).$$

This simplicial set fits into a tensor-hom adjunction given by

$$\mathrm{Hom}_{\mathrm{sSets}}(Z \times X, Y) \cong \mathrm{Hom}_{\mathrm{sSets}}(Z, \mathrm{map}_{\mathrm{sSets}}(X, Y)).$$

Indeed, from this adjunction we may deduce the formula for $\mathrm{map}(X, Y)_n$ by evaluating at $Z = \Delta^n$.

Abstracting these formulas, one arrives at the axioms for a **simplicial category** [3, II Definition 2.1]. A simplicial category is a category M equipped with

(1) a **mapping space functor**: $\mathrm{map} : M^{op} \times M \to \mathrm{sSets}$, written $\mathrm{map}_M(X, Y)$,
(2) an **action** of sSets, $M \times \mathrm{sSets} \to M$, written $X \otimes S$, and
(3) an **exponential**, $\mathrm{sSets}^{\mathrm{op}} \times M \to M$, written X^S for an object $X \in M$ and a simplicial set S

subject to certain compatibilities. The most important are that

$$- \otimes X : \mathrm{sSets} \leftrightarrows C : \mathrm{map}_M(X, -)$$

should be an adjoint pair of functors and that $\mathrm{Hom}_M(X, Y) \cong \mathrm{map}_M(X, Y)_0$ for all $X, Y \in M$.

Suppose that M is a simplicial category simultaneously equipped with a model structure. We would like the simplicial structure above to play well with the model structure. For example, if $i : A \to X$ is a cofibration, we expect $\mathrm{map}_M(Y, A) \to \mathrm{map}_M(Y, X)$ to be a fibration (and hence induce long exact sequences in homotopy groups) for any object Y as is the case in simplicial sets.

Definition 2.1. Suppose that M is a model category which is also a simplicial category. Then M satisfies **SM7**, and is called a **simplicial model category**, if for any cofibration $i : A \to X$ and any fibration: $p : E \to B$ the map of simplicial sets (induced by the functoriality of map)

$$\mathrm{map}_M(X, E) \to \mathrm{map}_M(A, E) \times_{\mathrm{map}_M(A,B)} \mathrm{map}_M(X, B)$$

is a fibration of simplicial sets which is moreover a weak equivalence if either i or p is.

Exercise 2.2. Show that in a simplicial model category M, if $A \to X$ is a fibration, then for any cofibrant object Y, the natural map $\mathrm{map}_M(Y, A) \to \mathrm{map}_M(Y, X)$ is a fibration of simplicial sets.

Another feature of simplicial model categories is the fact that one may define a concept of homotopy that is more transparent than in an ordinary model category (where one defines left and right homotopies, see [2]). Suppose that $A \in M$ is a cofibrant object, then we say that two morphisms $f, g : A \to X$ are homotopic if there is a morphism: $H : A \otimes \Delta^1 \to X$ such that

$$\begin{array}{ccc} A \coprod A & \xrightarrow{d_1 \coprod d_0} & A \otimes \Delta^1 \\ {\scriptstyle f \coprod g}\downarrow & \swarrow {\scriptstyle H} & \\ X & & \end{array}$$

commutes. Write $f \sim g$ if f and g are homotopic.

Exercise 2.3. Prove that $\sim$ is an equivalence relation on $\mathrm{Hom}_M(A, X)$ when A is cofibrant.

Exercise 2.4. Show that the equivalence relation is the same as $\sim_l$.

3. A taste of localization and rational homotopy theory

Definition 3.1. Let M be a simplicial model category with class of weak equivalences W. Suppose that I is a *set* of maps in M. An object X of M is **I-local** if it is fibrant and if for all $i : A \to B$ with $i \in I$, the induced morphism on mapping spaces $i^* : \mathrm{map}_M(B, X) \to \mathrm{map}_M(A, X)$ is a weak equivalence (of simplicial sets). A morphism $f : A \to B$ is an **I-local equivalence** if for every I-local object X, the induced morphism on mapping spaces $f^* : \mathrm{map}_M(B, X) \to \mathrm{map}_M(A, X)$ is a weak equivalence. Let W_I be the class of all I-local equivalences. By definition, $W \subseteq I$.

Let F_I denote the class of maps satisfying the right lifting property with respect to W_I-acyclic cofibrations ($W_I \cap C$). If (W_I, C, F_I) is a model category structure on M, we call this the **left Bousfield localization** of M with respect to I.

To distinguish between the model category structures on M, we will write $\mathrm{L}_I M$ for the left Bousfield model category structure on M. We will only write $\mathrm{L}_I M$ when the classes of morphisms defined above do define a model category structure.

When it exists, the Bousfield localization of M with respect to I is universal with respect to Quillen pairs $F : M \rightleftarrows N : G$ such that $\mathbf{L}F(i)$ is a weak equivalence in N for all $i \in I$.

Exercise 3.2. Show that if it exists, then the identity functors $\mathrm{id}_M : M \rightleftarrows M : \mathrm{id}_M$ induce a Quillen pair between M (on the left) and $\mathrm{L}_I M$.

We want to quote an important theorem asserting that in good cases the left Bousfield localization of a model category with respect to a set of morphisms exists. Some conditions, which we will not define, are needed on the model category.

Definition 3.3. A model category M is **left proper** if pushouts of weak equivalences along cofibrations are weak equivalences.

Theorem 3.4. *If M is a (left proper and combinatorial) simplicial model category and I is a set of morphisms in M, then the left Bousfield localization $\mathrm{L}_I M$ exists and inherits a simplicial model category structure from M.*

Exercise 3.5. Let $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ denote as usual the category of non-negatively graded chain complexes with the projective model category structure. Note that the cofibrations are the degree-wise monomorphisms with projective cokernels. Let $\mathrm{L}_{\mathbb{Q}}\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ be the the Bousfield localization at the class of rational homology equivalences, i.e., those maps $f : M \to N$ such that

$$\mathrm{H}_*(f) \otimes \mathbb{Q} : \mathrm{H}_*(M) \otimes \mathbb{Q} \to \mathrm{H}_*(N) \otimes \mathbb{Q}.$$

is an isomorphism. Prove that $\mathrm{L}_{\mathbb{Q}}\mathrm{Ch}_{\geq 0}(\mathbb{Z}) \to \mathrm{Ch}_{\geq 0}(\mathbb{Q})$ is derivable and a Quillen equivalence.

References

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