Rational Homotopy Theory - Lecture 14

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1. Kähler differentials

Let k be a commutative ring, and let R be a commutative k-algebra. The R-module of **Kähler differentials** of R over k, denoted by $\Omega_{R/k}$ or simply Ω_R if the base is clear from context, is the free R-module on symbols dx for every $x \in R$ modulo the relations $da = 0$ for $a \in k$ and $d(xy) - d(x)y - xd(y)$ for all pairs $x, y \in R$.

The module of Kähler differentials is functorial in R. This means that if $R \xrightarrow{f} S$ is a map of commutative k-algebras, then there is a natural map $\Omega_{R/k} \to \Omega_{S/k}$ of R-modules, where we view $\Omega_{S/k}$ as an R-module by forgetting. In fact, if $R \xrightarrow{f} S$ is a map of commutative k-algebras, there is an exact sequence

$$S \otimes_R \Omega_{R/k} \to \Omega_{S/k} \to \Omega_{S/R} \to 0.$$

Sometimes these facts are easier to prove using the following universal properties.

A k-**derivation** of R in an R-module M is a map $\phi : R \to M$ such that $\phi(a) = 0$ for $a \in k$ and $\phi(xy) = x\phi(y) + \phi(x)$. When the ground ring k is clear from context, we will call such a map simply a derivation. The set of k-derivations of R in M forms an R-module under addition, $\mathrm{Der}_k(R, M)$. The map $d : R \to \Omega_{R/k}$ is a derivation, and it in fact is the universal derivation in the following sense.

Lemma 1.1. *The functor* $\mathrm{Der}_k(R, M)$ *is representable by* $\Omega_{R/k}$. *That is, there is a natural isomorphism of functors* $\mathrm{Hom}_R(\Omega_{R/k}, M) \xrightarrow{d^*} \mathrm{Der}_k(R, M)$.

There is another universal property which I like even more, which gives less formulaic definition of a derivation. Let $(\mathrm{CAlg}_k)_{/R}$ be the category of commutative k-algebras with a fixed map to R, and let $\mathrm{Hom}_{k/R}$ denote the hom sets in this category. Given an R-module M, let $R \oplus M$ denote the commutative k-algebra with multiplication map given by

$$(r, m) \cdot (s, n) = (rs, rn + sm).$$

This is called the **trivial square-zero extension of** R **by** M because $(0, m) \cdot (0, n) = 0$ for all $m, n \in M$.

Exercise 1.2. Show that $\mathrm{Der}_k(R, M)$ is naturally isomorphic to $\mathrm{Hom}_{k/R}(R, R \oplus M)$.

It follows, that there is a natural isomorphism $\mathrm{Hom}_R(\Omega_{R/k}, M) \cong \mathrm{Hom}_{k/R}(R, R \oplus M)$ for R-modules M. Many properties of the Kähler differentials follow from universal considerations.

Example 1.3. Show that if K/k is a finite separable extension of fields, then $\Omega_{K/k} = 0$.

Example 1.4. On the other hand, show that $\Omega_{\mathbb{C}/\mathbb{R}}$ is an uncountably-generated $\mathbb{C}$-module.

Example 1.5. Let $R = k[x_1, \ldots, x_n]$. Then, $\Omega_{R/k}$ is a free R-module with basis dx_i for $1 \leq i \leq n$.

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2. The algebraic de Rham complex

Let R be a commutative k-algebra. Then, $\Omega^*_{R/k}$, the exterior algebra on $\Omega^1_{R/k} = \Omega_{R/k}$ is called the **algebraic de Rham complex** of R over k. It is a graded-commutative R-algebra. The de Rham cohomology of $X = \operatorname{Spec} R$ over k is defined as

$$\mathrm{H}^*_{\mathrm{dR}}(X/k) = \mathrm{H}^*(\Omega^*_{\mathrm{R}/k}).$$

Note that this is functorial in maps of schemes, varieties, or rings. Only rings will concern us below.

We come to an absolutely crucial distinction between characteristic p and characteristic 0.

Lemma 2.1. *Let k be a field of characteristic* 0*. Let $R = k[x_1, \ldots, x_n]$, with $X = \operatorname{Spec} R = \mathbb{A}^n_k$. Then, $\mathrm{H}^*_{\mathrm{dR}}(\mathbb{A}^n/k) = k$.*

Proof. I claim that

$$\Omega^*_{k[x_1,\ldots,x_{n-1}]} \otimes_k \Omega^*_{k[x_n]} \cong \Omega^*_{k[x_1,\ldots,x_n]}.$$

Note that $\Omega^j_{k[x_1,\ldots,x_{n-1}]}$ has rank $\binom{n-1}{j}$, from which the result follows from a rank count and the fact that

$$\binom{n-1}{j} + \binom{n-1}{j-1} = \binom{n}{j}.$$

Hence, it suffices by the Künneth formula to show that $\mathrm{H}^*_{\mathrm{dR}}(\mathbb{A}^1/k) = 0$. This is the cohomology of the complex

$$k[x] \xrightarrow{d} \Omega_{k[x]/k}.$$

By the fundamental theorem of calculus, this map is surjective. For example,

$$d\left(\frac{x^{n+1}}{n+1}\right) = x^n.$$

We see here why we need characteristic zero. On the other hand, if $d(f(x)) = 0$, then it follows that $f(x)$ is constant, as desired. ☐

Remark 2.2. Bousfield and Guggenhiem call this the algebraic Poincaré lemma for obvious reasons.

Remark 2.3. What happens when the characteristic of k is p? Then, $d(x^{pm}) = 0$ for all $m \geq 1$. It follows that

$$\mathrm{H}^0(\mathbb{A}^1/k) = k[x^p],$$

while $\mathrm{H}^1(\mathbb{A}^1/k)$ is a free $k[x^p]$-module on $x^{p-1}dx$.

3. The polynomial differential forms on the standard simplices

Fix a commutative ring k. Let $\Delta^\bullet_{\mathrm{alg}}$ be the cosimplicial k-scheme given by the algebraic simplices, so that

$$\Delta^n_{\mathrm{alg}} = \operatorname{Spec} k[x_0, \ldots, x_n]/(x_0 + \cdots + x_n - 1).$$

Note that $\Delta^\bullet$ is the affine scheme associated to a *simplicial* commutative k-algebra,

$$n \mapsto k[x_0, \ldots, x_n]/(x_0 + \cdots + x_n - 1).$$

One can take the algebraic de Rham complex to obtain the simplicial cdga $\Omega^*_{\mathbb{A}^\bullet/k}$. This is the simplicial cdga denoted by $\nabla(\bullet, *)$ in Bousfield and Guggenheim, where

$$\nabla(p, *) = \Omega^*_{\mathbb{A}^p/k}.$$

So, it is a cdga in the second variable, and a simplicial object in the first variable.

We saw above that $\nabla(p, *)$ is acyclic for each $p \geq 0$. Specifically, the natural map $\eta : k \to \nabla(p, *)$ is a quasi-isomorphism for each p. In fact, this map is a chain equivalence. As in the proof of Lemma 2.1, it suffices to prove this for $p = 1$. Recall that to construct a chain equivalence, besides η we must specify $\epsilon : \nabla(1, *) \to k$ such that $\epsilon \circ \eta = \mathrm{id}_k$ and a chain homotopy $h : \nabla(1, *) \to \nabla(1, *)$ such that $dh + hd = \mathrm{id}_{\nabla(1,0)} - \eta \circ \epsilon$. We let $\eta(f) = f(0)$ for $f \in \nabla(1, 0)$ and $\eta(\omega) = 0$ for $\omega \in \nabla(1, 1)$. Clearly, $\epsilon \circ \eta = \mathrm{id}_k$. Similarly, we set $h(f) = 0$

and if $\omega = (\sum_{i=0}^{n} a_i x^i)dx$ then $h(\omega) = \sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1}$. It is easy to verify that h is a chain homotopy from $\eta \circ \epsilon$ to $\mathrm{id}_{\nabla(1,0)}$.

It turns out that $\nabla(\bullet, q)$ is also simplicially contractible for each $q \geq 0$. There are a couple of ways to prove this. One is to give an explicit contracting homotopy, which is what Bousfield and Guggenheim do. We take a different approach. Note that $\nabla(\bullet, q)$ is a simplicial abelian group (or simplicial k-module), and as such it is a Kan complex. Moreover,

$$\pi_* |\nabla(\bullet, q)| \cong \mathrm{H}_*(\mathrm{N}\nabla(\bullet, q)),$$

where $\mathrm{N}\nabla(\bullet, q)$ is the chain complex associated to the simplicial k-module $\nabla(\bullet, q)$. Hence, to show that $\nabla(\bullet, q)$ is simplicially contractible, it suffices to show that $\mathrm{H}_*(\mathrm{N}\nabla(\bullet, q)) = 0$ for all $q \geq 0$. Note that $\nabla(\bullet, 0)$ is a simpicial k-algebra and that $\nabla(\bullet, q)$ is a simplicial module over $\nabla(\bullet, 0)$ in the obvious sense. In particular, these facts mean that $\mathrm{H}_*(\mathrm{N}\nabla(\bullet, 0))$ is a graded-commutative k-algebra and that $\mathrm{H}_*(\mathrm{N}\nabla(\bullet, q))$ is a graded module over this ring for $q \geq 0$. Hence, it suffices to show that $\mathrm{H}_*(\mathrm{N}\nabla(\bullet, 0)) = 0$. But, it is easy to see that $\mathrm{H}_0(\mathrm{N}\nabla(\bullet, 0)) = 0$, so the graded ring has $1 = 0$, so it is zero, as desired.

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