Rational Homotopy Theory - Lecture 16

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Basically, we discusses the same material in lecture on 10 March 2016 as well.

1. The PL de Rham theorem

We are going to take a slightly different approach, based on the presentation in Félix-Halperin-Thomas [3], with some category-theoretical improvements to make our lives easier.

Recall that we have the simplicial cdga $\nabla(\bullet, *)$, and the rational PL de Rham complex of a simplicial set X is

$$\mathrm{A}^*(X) = \mathrm{Hom}_{\mathrm{sSets}}(X, \nabla(\bullet, *)).$$

Now, given any simplicial dga $R(\bullet, *)$, we let

$$\mathrm{A}^*_R(X) = \mathrm{Hom}_{\mathrm{sSets}}(X, R(\bullet, *)).$$

So, as an example, we have $\mathrm{A}^*(X) = \mathrm{A}^*_\nabla(X)$. We call $\mathrm{A}^*_R(X)$ the **cochains on X with coefficients in R.**

We will introduce a simplicial dga N such that $\mathrm{A}^*_{\mathrm{N}}$ is naturally isomorphic to $\mathrm{N}^*(X)$, the normalized cochain algebra of X. In fact, let

$$\mathrm{N}(\bullet, q) = \mathrm{N}^q(\Delta^\bullet).$$

Lemma 1.1. *For any simplicial set X, the natural map $\mathrm{A}^*_{\mathrm{N}}(X) \to \mathrm{N}^*(X)$ is an isomorphism.*

Proof. Let $f \in \mathrm{A}^q_{\mathrm{N}}(X) = \mathrm{Hom}_{\mathrm{sSets}}(X, \mathrm{N}(\bullet, q))$. For a p-simplex τ of X, let $f_\tau \in \mathrm{N}(p, q)$ be the normalized q-cochain on Δ^p. Given a q-simplex $\sigma \in X_q$, we can apply f to obtain $f_\sigma = f(\sigma)e_q \in \mathrm{N}(q, q) = \mathbb{Q} \cdot e_q$, where $\hat{e}_q$ is dual to the fundamental simplex e_q of Δ^q. One checks that $\sigma \mapsto f(\sigma)$ defines an element of $\mathrm{N}^q(X)$, and that the assignment $\mathrm{A}^*_{\mathrm{N}}(X) \to \mathrm{N}^*(X)$ is a dga map. If f vanishes on all q-simplices, then it must vanish on all simplices of X. To see this, let $\tau : \Delta^p \to X$ be a p-simplex of X, and let $\alpha : \Delta^q \to \Delta^p$ be some composition of face and degeneracy maps. Since f is a simplicial map, $f_\tau(\alpha) = f_{\tau\circ\alpha}(e_q) = 0$.

Now, suppose that $F \in \mathrm{Hom}(X_q, \mathbb{Q})$ is a normalized cochain, so that $F(\sigma_i(\tau)) = 0$ for any i and $\tau \in X_{q-1}$. Let $\tau : \Delta^p \to X$, and define $f(\tau) = \mathrm{N}^p(F) \in \mathrm{N}^q(\Delta^p) = \mathrm{N}(p, q)$. Hence, $\mathrm{A}^*_{\mathrm{N}}(X) \to \mathrm{N}^*(X)$ is surjective. □

Theorem 1.2. *The natural maps*

$$\mathrm{A}^*_{\mathrm{N}}(X) \to \mathrm{A}^*_{\mathrm{N}\otimes\nabla}(X) \leftarrow \mathrm{A}^*_\nabla(X)$$

are quasi-isomorphisms of dgas for any simplicial set X.

We will need some more preliminaries before proving this. We call a simplicial dga $R(\bullet, *)$ **degree-wise contractible** if $R(\bullet, q)$ the simplicial abelian group is contractible for all q. Note that in Félix-Halperin-Thomas this property is called 'extendable'. But, we will just call it what it is.

Proposition 1.3. *Let $G_\bullet$ be a simplicial group. Then, $G_\bullet$ is fibrant as a simplicial set.*

Proof. Recall that in order to be fibrant, dotted lifts must exist in any solid-arrow diagram

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & G \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & *. \end{array}$$

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This is equivalent to the following condition: for any $x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in G_{n-1}$ such that $\partial_i x_j = \partial_{j-1} x_i$, $i < j$ and $i, j \neq k$, there exists $y \in G_n$ such that $\partial_i y = x_i$ for $i \neq k$. We construct a filling y inductively as follows. Let $g_{-1} = 1$, the identity element of G_n. Assume we have constructed g_{r-1} such that $\partial_i g_{r-1} = x_i$ for $0 \leq i \leq r-1$, $i \neq k$. If $r = k$, set $g_r = g_{r-1}$. Otherwise, if $r \neq k$, define $u = x_r^{-1} \partial_r(g_{r-1})$. If $i < r$,

$$\begin{aligned}\partial_i(u) &= \partial_i(x_r^{-1})\partial_i\partial_r g_{r-1} \\ &= (\partial_i x_r)^{-1}\partial_{r-1}\partial_i g_{r-1} \\ &= (\partial_i x_r)^{-1}\partial_{r-1} x_i \\ &= 1,\end{aligned}$$

by hypothesis on the x_i. Thus, if we set $g_r = g_{r-1}(\sigma_r u)^{-1}$, we have $\partial_i(g_r) = x_i(\sigma_{r-1}\partial_i(u))^{-1} = x_i$ if $i < r$, and $\partial_r(g_r) = \partial_r(g_{r-1})u^{-1} = x_r$. Thus, taking $y = g_n$ works. □

Remark 1.4. Xing Gu asked in class why this proof does not work to show that $G_\bullet$ satisfies the lifting property with respect to all diagrams

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & G \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & *. \end{array}$$

In other words, why doesn't the proof show moreover that $G_\bullet$ is contractible. The basic reason is as follows. If we took a sequence $x_0, \ldots, x_n \in G_n$ such that $\partial_i x_j = \partial_{j-1} x_i$ as in the proof, then the proof would work to construct g_{n-1} such that $\partial_i(g_{n-1}) = x_i$ for $0 \leq i \leq n-1$. What happens in degree n? We define $u = x_n^{-1}\partial_n(g_{n-1})$, and then we set $g_n = g_{n-1}(\sigma_n u)^{-1}$. All good, right? **Wrong!** The class u is an $n-1$-simplex, so there is no nth degeneracy map to apply to it! This is related to the fact that a connected simplicial set with an **extra degeneracy** is contractible. If we had an extra degeneracy, the proof would work.

Here are a couple remarks related to this question. Recall that if G is a group, BG is the simplicial set with $BG_n = G^n$ (so that $BG_0 = *$). The face maps are given by $\sigma_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n)$ and

$$\partial_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & \text{if } i = 0, \\ (g_1, \ldots, g_i g_{i+1}, g_{i+2}, \ldots, g_n) & \text{if } 0 < i < n, \\ (g_1, \ldots, g_{n-1}) & \text{if } i = n. \end{cases}$$

As mentioned before I think, BG is called the **classifying space** of G, and indeed we have $|BG|$ is a $K(G,1)$-space.

Exercise 1.5. Show that BG is a simplicial group if and only if G is abelian.

Exercise 1.6. Let A be an abelian group. Prove that by hand that if every diagram

$$\begin{array}{ccc} \partial\Delta^2 & \longrightarrow & BA \\ \downarrow & \nearrow & \downarrow \\ \Delta^2 & \longrightarrow & *. \end{array}$$

has a lift, then $A = 0$.

Lemma 1.7. *Suppose that $R(\bullet, *)$ is degree-wise contractible and that $X \subseteq Y$ is an inclusion of simplicial sets. Then, $\mathrm{A}^*_R(Y) \to \mathrm{A}^*_R(X)$ is surjective.*

Proof. Since $R(\bullet, q)$ is a Kan complex for all q, contractibility implies that $R(\bullet, q) \to *$ is an acyclic fibration. But, $X \to Y$ is a cofibration. It follows that there is always a lift in the diagram

$$\begin{array}{ccc} X & \longrightarrow & R(\bullet, q) \\ \downarrow & \nearrow & \downarrow \\ Y & \longrightarrow & *, \end{array}$$

for any q. This proves the lemma. ☐

Example 1.8. We saw that $\nabla(\bullet, *)$ is degree-wise contractible in Lecture 14.

Lemma 1.9. *The simplicial dga* N *is degree-wise contractible.*

By construction, $\mathrm{H}^*(\mathrm{N}(p,*))$ is the cellular $\mathbb{Q}$-cohomology of Δ^p_{top}, which is a contractible space, so it vanishes in positive degrees and is $\mathbb{Q}$ in degree 0.

Proof. Consider $\mathrm{N}(\bullet, q)$. As in the argument for the contractibility of $\nabla(\bullet, q)$, it is enough to consider the $q = 0$ case, since it is enough to show that the homology of $\mathrm{N}(\bullet, q)$ vanishes, and this is a graded module over the graded ring $\mathrm{N}(\bullet, 0)$. Now, consider $\mathrm{N}(1,0) \rightrightarrows \mathrm{N}(0,0)$. Note that $\mathrm{N}(p,0) = \mathrm{Hom}(\Delta^p_0, \mathbb{Z}) \cong \mathbb{Q}^{p+1}$. With the natural basis, $\mathrm{N}(1,0) \rightrightarrows \mathrm{N}(0,0)$ is $\mathbb{Q}^2 \rightrightarrows \mathbb{Q}$. The chain complex associated to $\mathrm{N}(\bullet, 0)$ has lowest differential $\partial_0 - \partial_1 : \mathbb{Q}^2 \to \mathbb{Q}$, which we can write in matrix form as $\begin{pmatrix} -1 & 1 \end{pmatrix}$. Evidently this is surjective, so that there is no degree zero homology. Since $\mathrm{H}_*\mathrm{N}(\bullet, 0)$ has a ring structure via the Alexander-Whitney map, and since $1 = 0$ in this ring, we have that the ring is zero, as desired. ☐

Given a pair $Y \subseteq X$ and a degree-wise contractible simplicial dga R, we define $\mathrm{A}^*_R(X, Y)$ to be the kernel of $\mathrm{A}^*_R(X) \to \mathrm{A}^*_R(Y)$. These are the **cochains of the pair with coefficients in** R.

Proposition 1.10. *If* $R \to S$ *is a map of degree-wise contractible simplicial dgas such that* $R(p,*) \to S(p,*)$ *is a quasi-isomorphism for all* $p \geq 0$*, then* $\mathrm{A}^*_R(X,Y) \to \mathrm{A}^*_S(X,Y)$ *is a quasi-isomorphism for all pairs* $Y \subseteq X$.

Proof. It is enough to prove the proposition for $Y = \emptyset$, so that we just have to prove that $\mathrm{A}^*_R(X) \to \mathrm{A}^*_S(X)$ is a quasi-isomorphism for all simplicial sets X. Note that $\mathrm{A}^*_R(\Delta^p) \cong R(p,*)$ and $\mathrm{A}^*_S(\Delta^p) \cong S(p,*)$, by representability. Let $\mathrm{sk}_n X$ be the n-skeleton of X. Note that $\mathrm{sk}_0 X$ is the disjoint union of the 0-simplices of X. Since this is a coproduct,

$$\coprod_{\Delta^0 \to X} \Delta^0,$$

it follows from our hypothesis that $\mathrm{A}^*_R(\mathrm{sk}_0 X) \to \mathrm{A}^*_S(\mathrm{sk}_0 X)$ is a quasi-isomorphism. We prove by induction that if the claim is true for all $p-1$-dimensional simplicial sets, then it is true for all n-dimensional simplicial sets. So, assume that $p - 1 \geq 0$ and that $\mathrm{A}^*_R(\mathrm{sk}_{p-1}X) \to \mathrm{A}^*_S(\mathrm{sk}_{p-1}X)$ is a quasi-isomorphism for all simplicial sets X. Note that this includes the boundary $\partial\Delta^p$. Since we know that we get a quasi-isomorphism for Δ^p, this implies that all three vertical maps are quasi-isomorphisms in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{A}^*_R(\Delta^p, \partial\Delta^p) & \longrightarrow & \mathrm{A}^*_R(\Delta^p) & \longrightarrow & \mathrm{A}^*_R(\partial\Delta^p) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{A}^*_S(\Delta^p, \partial\Delta^p) & \longrightarrow & \mathrm{A}^*_S(\Delta^p) & \longrightarrow & \mathrm{A}^*_S(\partial\Delta^p) & \longrightarrow & 0. \end{array}$$

Suppose that Y is $p-1$-dimensional, and that X is obtained from Y by adding a single non-degenerate p-simplex σ. Note that in this case, the boundary of σ is contained in Y. In this case, $\mathrm{A}^*_R(X,Y) \cong \mathrm{A}^*_R(\Delta^p, \partial\Delta^p)$, and similarly for S. Indeed, both sides are completely determined by where they send the unique p-simplex not in Y or $\partial\Delta^p$, respectively. It follows that $\mathrm{A}^*_R(\mathrm{sk}_p X) \to \mathrm{A}^*_S(\mathrm{sk}_p X)$ is a (possibly transfinite) filtered limit of quasi-isomorphisms, and hence it is a quasi-isomorphism by the lemma below when I is sufficiently small. Since $X = \mathrm{colim}_p\, \mathrm{sk}_p X$, we again have $\mathrm{A}^*_R(X) = \lim_p \mathrm{A}^*_R(\mathrm{sk}_p X)$, the next lemma works for X since $\mathbb{N}$ is $\aleph_1$-small. In the general case for going from $\mathrm{sk}_{p-1}X$ to $\mathrm{sk}_p X$, it is better to argue that $\mathrm{A}_R^*(\mathrm{sk}_p X, \mathrm{sk}_{p-1} X) \cong \bigoplus \mathrm{A}_R^*(\Delta^p, \partial\Delta^p)$ where the direct sum is over all non-degenerate p-simplices of X. ☐

Lemma 1.11. *Suppose that I is an $\aleph_\omega$-small filtered category, and let $F, G : I^{\mathrm{op}} \to \mathrm{Ch}^{\geq 0}$ be functors from I^{op} to non-negatively graded cochain complexes with a natural transformation $F \to G$. If $F(i) \to G(i)$ is a quasi-isomorphism for all $i \in I$, then $\lim_{I^{\mathrm{op}}} F(i) \to \lim_{I^{\mathrm{op}}} G(i)$ is a quasi-isomorphism.*

Proof. Since I is small and filtered, the derived functors $\mathrm{R}^p \lim$ vanish for $p >> 0$ by work of Jensen (1970). It follows that the spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{R}^p \lim_i \mathrm{H}^q(F(i)) \Rightarrow \mathrm{H}^{p+q}(\lim_i F(i))$$

converges, from which the lemma follows from the functoriality of spectral sequences. ☐

Question 1.12. Can we prove the lemma in full generality for small filtered I using homotopy limits and model categories?

Proof of Theorem 1.2. We can apply Proposition 1.10 to the two morphisms $\mathrm{N} \to \mathrm{N} \otimes \nabla \leftarrow \nabla$. We only have to observe that $\mathrm{N} \otimes \nabla$ is degree-wise contractible. In degree q, we have

$$(\mathrm{N} \otimes \nabla)(\bullet, q) \cong \bigoplus_{a+b=q} \mathrm{N}(\bullet, a) \otimes \nabla(\bullet, b).$$

The homology of each summand on the right side vanishes by Künneth. ☐

What's very nice about this approach is that we get multiplicativity without further work, and this answers Thom's question completely.

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