Rational Homotopy Theory - Lecture 22

BENJAMIN ANTIEAU

1. Sketch of proof of the Sullivan-de Rham equivalence

Suppose that $F : C \rightleftarrows D : G$ is a pair of adjoint functors, and suppose that F and G restrict to adjoint functors $F' : C' \rightleftarrows D' : G'$. We can ask when F' and G' are inverse equivalences. Given an object $c \in C$ there is a unit map

$$c \to GFc,$$

and for $d \in D$ there is a counit map

$$FGd \to d.$$

Exercise 1.1. Show that $F' : C \rightleftarrows D' : G'$ is a pair of inverse equivalences if and only if the unit and counit maps are all isomorphisms.

Theorem 1.2 (The Sullivan-de Rham theorem)**.** *The derived functors*

$$\mathbf{L}\mathrm{F} : \mathrm{Ho}(\mathrm{cdga}^{\geq 0}_{\mathbb{Q}}) \rightleftarrows \mathrm{Ho}(\mathrm{sSets}^{\mathrm{op}}) : \mathbf{R}\mathrm{A}$$

restrict to inverse equivalences

$$\mathrm{Ho}(\mathrm{cdga}^{\geq 0}_{\mathbb{Q}})^{\mathrm{f}} \rightleftarrows \mathrm{Ho}(\mathrm{sSets}^{\mathrm{op}})^{\mathrm{fN}\mathbb{Q}}.$$

Similarly, the derived functors

$$\mathbf{L}\mathrm{F} : \mathrm{Ho}(\mathrm{cdga}^{\geq 0}_{/\mathbb{Q}}) \rightleftarrows \mathrm{Ho}(\mathrm{sSets}^{\mathrm{op}}_{*}) : \mathbf{R}\mathrm{A}$$

restrict to inverse equivalences

$$\mathrm{Ho}(\mathrm{cdga}^{\geq 0}_{/\mathbb{Q}})^{\mathrm{f}} \rightleftarrows \mathrm{Ho}(\mathrm{sSets}^{\mathrm{op}}_{*})^{\mathrm{fN}\mathbb{Q}}.$$

Exercise 1.3. Show that the unpointed version of the theorem follows from the pointed version.

Outline of proof. The exercises above reduce the problem to show that the unit and counit maps are isomorphisms for *augmented* cdgas of finite $\mathbb{Q}$-type and pointed rational nilpotent spaces of finite $\mathbb{Q}$-type and that $\mathbf{L}\mathrm{F}$ and $\mathbf{R}\mathrm{A}$ restrict to the specified subcategories. Thus, we have to show the following

(1) the unit map $X \to \mathbf{R}\mathrm{A}\mathbf{L}\mathrm{F}(X)$ is an isomorphism in the homotopy category when X is an augmented cdga of finite $\mathbb{Q}$-type and $\mathbf{L}\mathrm{F}(X)$ is a connected rational nilpotent space of finite $\mathbb{Q}$-type;
(2) the counit map $\mathbf{L}\mathrm{F}\mathbf{R}\mathrm{A}(Y) \to Y$ is an isomorphism in the homotopy category (of $\mathrm{sSets}^{\mathrm{op}}_{*}$) when X is a connected rational nilpotent space of finite $\mathbb{Q}$-type and $\mathbf{R}\mathrm{A}(Y)$ is an augmented cdga of finite $\mathbb{Q}$-type.

We make a couple additional simplifying observations. Since every augmented connected rational cdga is quasi-isomorphic to a minimal cdga, we can assume that X is minimal in (1), in which case, $\mathbf{L}\mathrm{F}(X)(X) \cong \mathrm{F}(X)$. Moreover, since every object of $\mathrm{sSets}^{\mathrm{op}}_{*}$ is fibrant, $\mathbf{R}\mathrm{A}(Y) \cong Y$ for all Y. So, in (1) it suffices to prove that $X \to \mathrm{AF}(X)$ is a quasi-isomorphism for X minimal.

In (2), we find it more convenient to write the arrow $Y \to \mathbf{L}\mathrm{F}\mathbf{R}\mathrm{A}(Y)$ in the doubly-opposite category, i.e., the category of rational nilpotent spaces of finite $\mathbb{Q}$-type. Since as

Date: 7 April 2016.

mentioned above Y is automatically cofibrant, we can write this arrow as $Y \to \mathrm{F}(\mathrm{MA}(Y))$, where $\mathrm{MA}(Y) \to \mathrm{A}(Y)$ is a minimal model. □

We define minimal algebras $W(n)$ for $n \geq 1$ as follows. If n is odd, set $W(n) = S(n)$, the free graded-commutative cdga on a degree n class x with $dx = 0$. If n is even, let $W(n) = \mathbb{Q}[x_n, y_{2n-1}]$ with $d(x_n) = 0$ and $d(y_{2n-1}) = x^2$. Hence, $W(n) \to \mathrm{A}(S^n)$ is a minimal model for all $n \geq 1$.

Proposition 1.4. *If X is a cofibrant cdga, then $\pi_n \mathrm{F}(X) \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^n X, \mathbb{Q})$, and if X is coconnective, then $\mathrm{F}(X)$ is connected. Moreover, the maps $\pi_n \mathrm{F}(X) \to \mathrm{Hom}_{\mathbb{Q}}(\pi^n X, \mathbb{Q})$ are group homomorphisms for $n \geq 2$. Finally, $\pi_1 \mathrm{F}(X[1,1]) \to \mathrm{F}(X)$ is surjective.*

Proof. We prove the first statement, leaving the rest as a difficult exercise. We consider the following composition using the Quillen adjunctions at the level of homotopy categories:

$$\pi_n \mathrm{F}(X) = [S^n, \mathrm{F}(X)] \cong [X, \mathrm{A}(S^n)] \cong [X, W(n)] \cong [X, V(n)] \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^n X, \mathbb{Q}),$$

where the final isomorphism follows from Lemma 1.1 of Lecture 19. Here we use a fixed quasi-isomorphism $W(n) \to V(n)$, which is itself a minimal model of $V(n)$. □

Lemma 1.5. *Statement (2) holds for $Y = K(\mathbb{Q}, n)$, $n \geq 1$.*

Proof. Note that Y is indeed a rational nilpotent space of finite $\mathbb{Q}$-type. Since we know the rational cohomology of $K(\mathbb{Q}, n)$, we can find a minimal model, which turns out to be $S(n)$. Hence, we have a quasi-isomorphism $S(n) \to \mathrm{A}(Y)$. We get an isomorphism $\mathrm{F}(S(n)) \to \mathbf{L}\mathrm{FA}(Y)$, and we must show these spaces are $K(\mathbb{Q}, n)s$. But, by the proposition, we see that $\mathrm{F}(S(n))$ is a $K(\mathbb{Q}, n)$-space. It suffices to show that the counit map is an equivalence. This follows because $\mathrm{AFA}(X) \to \mathrm{A}(X)$ has a right inverse. □

Lemma 1.6. *Statement (1) holds for $X = S(n)$, $n \geq 1$, the minimal cdga on a degree n generator x with $dx = 0$.*

Proof. Indeed, we saw in the previous lemma that $\mathrm{F}(S(n))$ is a $K(\mathbb{Q}, n)$-space, and hence a minimal model of $\mathrm{AF}(S(n))$ is given by $S(n)$ itself. Once again, $\mathrm{FAF}(S(n)) \to \mathrm{F}(S(n))$ has a right inverse, so $\pi^* S(n) \to \mathrm{AF}(S(n))$ is injective and hence an isomorphism. □

References

[1] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994.

[2] A. K. Bousfield and V. K. A. M. Gugenheim, *On* PL *de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **8** (1976), no. 179, ix+94.

[3] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126.

[4] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001.

[5] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999.

[6] P. Goerss and K. Schemmerhorn, *Model categories and simplicial methods*, Interactions between homotopy theory and algebra, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, pp. 3–49.

[7] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin-New York, 1967.

[8] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.