Rational Homotopy Theory - Lecture 23

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1. The formality result of DGMS

A cdga X is **formal** if it is quasi-isomorphic to a cdga Y with 0-differential. Of course, $Y \cong (\mathrm{H}^*(X), 0)$, the cohomology ring of X equipped with the zero differential. By quasi-isomorphic, we mean that X and $\mathrm{H}^*(X)$ are isomorphic in the homotopy category of cdgas. The definition in [2] is slightly different. They define formality only for minimal cdgas. For the moment, say that a minimal cdga M is **DGMS-formal** if there is a quasi-isomorphism of cdgas $M \to \mathrm{H}^*(M)$, where $\mathrm{H}^*(M)$ has the zero differential.

Remark 1.1. Note that we are using quasi-isomorphic and quasi-isomorphism in slightly different way. A quasi-isomorphism $X \to Y$ is in particular a map of cdgas from $X \to Y$, not just a map in the homotopy category. On the other hand, two cdgas are quasi-isomorphic if there is a zig-zag of quasi-isomorphisms between them.

Lemma 1.2. *A coconnective cdga X is formal if and only if its minimal model M is DGMS-formal.*

Proof. One direction is clear: if M is DGMS-formal, the zig-zag $X \leftarrow M \to \mathrm{H}^*(M) \cong \mathrm{H}^*(X)$ exhibits a quasi-isomorphism from X to $\mathrm{H}^*(X)$. So, suppose that X is formal. It is enough to prove that if $X \leftarrow Y \to \mathrm{H}^*(X)$ is a zig-zag of quasi-isomorphisms, then we can construct a dotted arrow making the following solid-arrow diagram commute:

$$\begin{array}{ccc} & Y & \longrightarrow \mathrm{H}^*(X) \\ & \downarrow & \nearrow \\ M & \longrightarrow & X. \end{array}$$

Since M is minimal, it is cofibrant. Hence, if $Y \to X$ were a (necessarily acyclic) fibration, such a lifting would exist automatically. What we can do instead is construct a minimal model N for Y. Of course, N maps directly to $\mathrm{H}^*(X)$ making it DGMS-formal. However, N is also a minimal model for X and it is hence isomorphic to M. This completes the proof. ☐

Deligne, Griffiths, Morgan, and Sullivan say that **the real homotopy type of a manifold M is a formal consequence of its cohomology** if $\mathrm{A}^*_{\mathrm{dR}}(M)$ is formal. There is an analogous notion for the complex homotopy type. Similarly, if X is a simplicial complex or simplicial set, the rational homotopy type of X is a formal consequence of its cohomology of $\mathrm{A}^*(X)$ is formal.

There is a similar notion for maps. A map of cdgas $f : X \to Y$ is **formal** if X and Y are formal, and if the induced map $M_f : M_X \to M_Y$ on minimal models fits into a *homotopy* commutative diagram

$$\begin{array}{ccc} M_X & \longrightarrow & M_Y \\ \downarrow & & \downarrow \\ \mathrm{H}^*(X) & \longrightarrow & \mathrm{H}^*(Y). \end{array}$$

We will not prove much about formal maps, but we will state the results on this topic in [2]. For the most part, the proofs are easy exercises once formality is known for X and Y.

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Remark 1.3. Note that everything we proved about minimal models and the model category structure on $\mathrm{cdga}_{\mathbb{Q}}^{\geq 0}$ works over $\mathbb{R}$ and $\mathbb{C}$. What doesn't hold is the Sullivan-de Rham equivalence. Nevertheless, one can construct real or complex cdgas $\mathrm{A}^*(X,\mathbb{R})$ or $\mathrm{A}^*(X,\mathbb{C})$ for any simplicial set X.

Formality is a somewhat strange condition. Let M be a manifold, and suppose that $\mathrm{A}^*_{\mathrm{dR}}(M)$ is formal. What does this really mean? For any manifold M we can find a subalgebra X of $\mathrm{A}^*_{\mathrm{dR}}(M)$ of differential forms such that the induced map $\mathrm{H}^*(X) \to \mathrm{H}^*_{\mathrm{dR}}(M)$ is a surjection. That is, we can find a compatible system of closed differential forms generating the cohomology of M. What we *cannot* do in general is choose the forms in such a way that the relations hold in the subalgebra X. For example, let α, β be two closed forms on M such that $[\alpha] \cup [\beta] = [\alpha \wedge \beta] = 0$ in the cohomology of M. Of course, this does not mean that $\alpha \wedge \beta = 0$. It only means that $\alpha \wedge \beta = d\gamma$ for some differential form γ. For formal manifolds, we can make such a choice.

Theorem 1.4 ([2])**.** *The real homotopy type of a compact Kähler manifold is a formal consequence of its cohomology. Similarly, if $f : X \to Y$ is a map between compact Kähler manifolds, then the induced map $f^* : \mathrm{A}^*_{\mathrm{dR}}(Y) \to \mathrm{A}^*_{\mathrm{dR}}(X)$ is a formal consequence of its cohomology.*

Remark 1.5. The theorem of DGMS holds in somewhat greater generality, namely for manifolds satisfying the so-called dd^c-lemma, which is discussed below. This class of manifolds includes for example complex Moishezon manifolds.

Why might we expect this for compact Kähler manifolds? DGMS give two motivations, one from Hodge theory and the other from the Weil conjectures. I'll just talk about the former. Recall that the Hodge theorem gives a decomposition

$$\mathrm{H}^m(X,\mathbb{C}) \cong_{p+q=m} \mathrm{H}^q(X,\Omega^p),$$

where Ω^p is the sheaf of holomorphic p-forms on X. We let $\mathrm{H}^{p,q}(X) = \mathrm{H}^q(X,\Omega^p)$. In particular, $\mathrm{H}^{p,0} = \mathrm{H}^0(X,\Omega^p)$ is the space of holomorphic p-forms. One result of Hodge theory is that all such forms are harmonic. This has the pleasing result that if α and β are holomorphic differential forms on X, then so is $\alpha \wedge \beta$, and moreover, $[\alpha \wedge \beta] = 0$ if and only if $\alpha \wedge \beta = 0$ in this case. Hence, under the Káhler hypothesis, the problems from the paragraph before the statement of Theorem 1.4 do not manifest themselves for holomorphic forms, the sub-algebra $\mathrm{H}^{*,0}(X)$ of $\mathrm{H}^*(X,\mathbb{C})$.

2. A little Hodge theory

Hodge theory on compact Kähler manifolds is a mixture of Hodge theory on compact orientable Riemannian manifolds, which we now describe, mixed with the complex structure. Recall that the Poincaré lemma says that the complex $0 \to \mathbb{C} \to \mathscr{A}^0 \xrightarrow{d} \mathscr{A}^1 \to \cdots \to \mathscr{A}^n \to 0$ is an exact complex of sheaves on an n-dimensional manifold M. Moreover, the sheaves $\mathscr{A}^q$ are all acyclic (specifically fine), as one sees by using partitions of unity. This means that one can compute $\mathrm{H}^*(X,\mathbb{C})$ as the cohomology of the complex

$$0 \to \mathrm{A}^0(M) \xrightarrow{\mathrm{A}}{}^1 (M) \to \cdots \to \mathrm{A}^d(M) \to 0.$$

That is, $\mathrm{H}^q(X,\mathbb{C}) \cong \mathrm{H}^q_{\mathrm{dR}}(X)$. Hodge theory takes this isomorphism one step farther by finding distinguished differential forms representing each cohomology class. These are the **harmonic** differential forms.

Recall that on a Riemannian manifold there is a $*$-operator, which assigns to each q-form on M an $n-q$-form. This is the global operation induced by integration and by the Riemannian metric which in particular induces an isomorphism between $\Lambda^q T^*_{\mathbb{C}}$ and $\Lambda^{n-q} T^*_{\mathbb{C}}$ on M. The adjoint of differentiation with respect to this pairing is $\delta = \pm * d*$, and the Laplacian is $\Delta = d\delta + \delta d$. The first Hodge theorem says that the kernel $\mathscr{H}^q(M)$ of the Laplacian $\ker(\Delta)$ on $\mathrm{A}^q(M)$ is a finite-dimensional space of closed forms and that the induced map $\mathscr{H}^q(M) \to \mathrm{H}^q(M)$ is an isomorphism for each q.

To prove this, note that $\Delta(\psi) = 0$ if and only if $d\psi = 0$ and $\delta\psi = 0$. Of course, one direction is clear. So, suppose that $\Delta(\psi) = 0$. Then,

$$0 = \langle \Delta(\psi), \psi \rangle = \langle d\delta\psi, \psi \rangle + \langle \delta d\psi, \psi \rangle = \langle \delta\psi, \delta\psi \rangle + \langle d\psi, d\psi \rangle.$$

Since the pairing is positive definite, this means that $d\psi = 0$ and $\delta\psi = 0$, as desired. In particular, every harmonic form is closed.

Spectral theory shows that $\mathscr{H}^q(M)$ is finite dimensional. The identity on $\mathrm{A}^q(M)$ decomposes as $\mathrm{id} = \mathscr{H} + \Delta \circ G$ where G is a so-called Green's operator, and where $\mathscr{H}$ is projection onto the harmonic forms. Let ψ be a closed i-form. Then, $\psi = \mathscr{H}(\psi) + \Delta(G(\psi))$. Now, G commutes with δ and d, which means that $\Delta(G(\psi)) = d\delta(G(\psi)) + \delta d(G(\psi)) = d\delta(G(\psi))$. This shows that every closed form is equal to a harmonic form in cohomology. Suppose that $d\psi$ is harmonic. Then, $\langle d\psi, d\psi \rangle = \langle \psi, \delta d\psi \rangle = 0$ because $\delta d\psi = 0$.

References

[1] A. K. Bousfield and V. K. A. M. Gugenheim, *On* PL *de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **8** (1976), no. 179, ix+94.

[2] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), no. 3, 245–274.