Rational Homotopy Theory - Lecture 24

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1. A little complex Hodge theory

Before proving the DGMS theorem, we give a little more background on Hodge theory, this time focusing on the complex case. Recall that a complex manifold is a smooth manifold M with an integrable almost complex structure on its *real* tangent bundle TM. An almost complex structure is an endomorphism J of TM such that $J^2 = -1$. This induces a splitting of the complex tangent bundle $T_{\mathbb{C}}M$ into $\pm i$-eigenspaces: $T_{\mathbb{C}}M = T'(M) \oplus T''(M)$, and $T'(M) \cong TM$ canonically.

Dualizing, one obtains a decomposition of the exterior powers of the cotangent bundle

$$\Lambda^r T^*_{\mathbb{C}}M = \bigoplus_{p+q=r} \Lambda^p T'(M)^* \otimes \Lambda^q T''(M)^*.$$

Writing $d = \partial + \overline{\partial}$, the almost complex structure is *integrable* (and hence M is a complex manifold) if $\overline{\partial}^2 = 0$ on the de Rham complex.

The Hodge theory developed for C^∞-forms works similarly for complex forms. Let's explain this very briefly.

Let $\mathscr{A}^{p,q}$ be the sheaf of differential $p+q$-forms of type p, q. The $\overline{\partial}$-Poincaré lemma says that for each p, the sequence

$$0 \to \Omega^p \to \mathscr{A}^{p,0} \xrightarrow{\overline{\partial}} \mathscr{A}^{p,1} \to \cdots \to \mathscr{A}^{p,n-p} \to 0$$

is a resolution of the sheaf of holomorphic p-forms Ω^p by acyclic sheaves. Hence, $\mathrm{H}^q(X, \Omega^p) \cong \mathrm{H}^{p,q}_{\overline{\partial}}(M)$, where

$$\mathrm{H}^{p,q}_{\overline{\partial}}(M)$$

is the qth cohomology of

$$0 \to \mathrm{A}^{p,0}(M) \xrightarrow{\overline{\partial}} \mathrm{A}^{p,1}(M) \to \cdots \to \mathrm{A}^{p,n-p}(M) \to 0.$$

Now, the differential d on is the total differential of the double complex involving the $\mathrm{A}^{p,q}(X)$, with a sign on the vertical differentials ∂. It follows that there is a spectral sequence

$$\mathrm{E}_1^{p,q} = \mathrm{H}^{p,q}_{\overline{\partial}}(M),$$

with differential d_1 induced by $\partial : \mathrm{H}^{p,q}_{\overline{\partial}}(M) \to \mathrm{H}^{p+1,q}_{\overline{\partial}}(M)$. Moreover, each of the cohomology classes in $\mathrm{H}^{p,q}_{\overline{\partial}}(M)$ is represented by a $\Delta_{\overline{\partial}}$-harmonic form, just as in the real Hodge theorem.

For a general compact complex manifold, there is no reason for this spectral sequence to degenerate, and indeed there are counterexamples. For a Kähler manifold however, the spectral sequence degenerates at E_1, and the Hodge decomposition then says that

$$\mathrm{H}^n_{\mathrm{dR}}(M, \mathbb{C}) \cong \bigoplus_{p+q=n} \mathrm{H}^q(X, \Omega^p).$$

The Kähler condition is somewhat technical, and even more mysterious. Any complex manifold M admits Hermitian metric on TM. These are bilinear pairings with $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$. Decomposing into real and complex parts, the real part is symmetric and the imaginary part is alternating. Hence, the imaginary part is represented by a $(1,1)$-form ω, and the Káhler condition is that $d\omega = 0$.

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We set $d^c = i(\overline{\partial} - \partial)$. Then, $d^c = J^{-1}dJ$, so that d^c is real, and $(d^c)^2 = 0$. This means that d^c respects $\mathrm{A}^*_{\mathrm{dR}}(M,\mathbb{R}) \subseteq \mathrm{A}^*_{\mathrm{dR}}(M,\mathbb{C})$. Additionaly, $dd^c = 2i\partial\overline{\partial} = -d^c d$. Moreover, the equation $d^c = J^{-1}dJ$ implies that $\mathrm{A}^*_{\mathrm{dR}}(M,\mathbb{R})$ is a cdga with differential d^c as well.

Lemma 1.1 (The dd^c Lemma)**.** *Let M be a compact Kähler manifold. If ψ is a differential form which is d-closed, d^c-closed, and a either a d-boundary or a d^c-boundary, then $\psi = dd^c\beta$ for some β.*

2. The first DGMS proof

DGMS give two proofs of their theorem, but we will only cover the first of those proofs. Consider the following diagram of real cdgas

$$\mathrm{A}^*_{\mathrm{dR}}(M,\mathbb{R}) \leftarrow \mathrm{Z}^*_{d^c}(M,\mathbb{R}) \rightarrow \mathrm{H}^*_{d^c}(M,\mathbb{R}) \tag{1}$$

where the left term is the de Rham complex, the middle term is the complex of d^c-closed forms with the de Rham differential d, and the right term is the cohomology of d^c-de Rham complex, but viewed still as a cdga with differential the induced differential d.

There are three things to do to check that this is well-defined. First, we need to know that d takes d^c-closed forms to d^c-closed forms, which follows from $dd^c = -d^c d$. Second, we need to know that d induces a differential on $\mathrm{H}^*_{d^c}(M,\mathbb{R})$. In other words, we need to know that $d(\psi + d^c\gamma) = d(\psi)$ modulo d^c-boundaries. But, $d(\psi + d^c\gamma) = d(\psi) + dd^c(\gamma) = d(\psi) - d^c d(\gamma)$, as desired. Third, we need to know that $\mathrm{Z}^*_{d^c}(M,\mathbb{R}) \rightarrow \mathrm{H}^*_{d^c}(M,\mathbb{R})$ is a morphism of cdgas, which is also clear. Another way of putting this is that the d^c-exact forms are a d-differential ideal in $\mathrm{Z}^*_{d^c}(M,\mathbb{R})$. But, it's clear that the d^c-exact forms are closed under d-differentiation, and if ψ is d^c-closed, then $\psi d^c\gamma = d^c(\psi\gamma)$.

Now, we claim that $\mathrm{H}^*_{d^c}(M,\mathbb{R})$ is formal. Suppose that ψ is a d^c-closed q-form, and consider $d\psi$, a d^c-closed $q+1$-form. We must show that $d\psi = d^c\lambda$. But, $d\psi$ is d^c-closed, d-closed, and a d-boundary. Hence, $d\psi = dd^c\beta = -d^c d\beta$, as desired.

If we prove that the maps above are quasi-isomorphisms, we will be done, having shown that $\mathrm{A}^*_{\mathrm{dR}}(M,\mathbb{R})$ is quasi-isomorphic to the formal cdga $\mathrm{H}^*_{d^c}(M,\mathbb{R})$. Let ψ represent a cohomology class in $\mathrm{H}^*_{d^c}(M,\mathbb{R})$, so that $d^c\psi = 0$ and $d\psi = d^c\lambda$ for some λ. Now, $d\psi$ is d and d^c-closed and is obvsiouly a d-boundary. Hence, $d\psi = dd^c\beta$ for some β. Thus, $\psi - d^c\beta$ is d-closed, and is also d^c-closed by inspection. It follows that $\mathrm{Z}^*_{\mathrm{dR}}(M,\mathbb{R}) \rightarrow \mathrm{H}^*_{d^c}(M,\mathbb{R})$ is surjective on cohomology.

Suppose that ψ is d^c-closed and d-closed, so that it represents a cohomology class of $\mathrm{Z}^*_{d^c}(M,\mathbb{R})$. If $[\psi]$ maps to zero in $\mathrm{H}^*_{d^c}(M,\mathbb{R})$, then $\psi = d^c\gamma$ for some γ. By the dd^c-lemma, $\psi = dd^c\beta$, so that $[\psi] = 0$.

Now we show that the left arrow is a quasi-isomorphism. Let ψ be a d-closed differential q-form. Then, $d^c\psi$ satisfies the dd^c-lemma, so $d^c\psi = dd^c\beta$. Hence, $\psi + d\beta$ is a d^c-closed form in the same cohomology class as ψ. That is, the left arrow is a surjection in cohomology.

Finally, suppose that ψ is a d-closed and d^c-closed q-form, so that $[\psi] \in \mathrm{H}^q(\mathrm{A}^*_{d^c}(M,\mathbb{R}))$ is a cohomology class. Assume that $\psi = d\gamma$ where γ is a differential $q-1$-form. We have no reason to think that γ is d^c-closed. But, ψ satisfies the dd^c-lemma, so that $\gamma = dd^c\beta$. But, $d^c(d^c\beta) = 0$, so ψ is exact in $\mathrm{Z}^*_{d^c}(M,\mathbb{R})$.

Because the algebras maps in (1) are functorial in the complex structure, it follows that maps between compact Kähler manifolds are formal.

3. Remarks

To what extent does the previous result use the theory that we have developed? On the one hand, the proof of the formality theorem itself uses none of the homotopy theory of cdgas. It relies instead on the identification of a particular quasi-isomorphic sub-cdga of $\mathrm{A}^*_{\mathrm{dR}}(M,\mathbb{R})$ which has an obvious quasi-isomorphism to a formal cdga. However, none of these cdgas is minimal. So, the fact that this is enough to conclude in some kind of natural way that the minimal model for $\mathrm{A}^*_{\mathrm{dR}}(M,\mathbb{R})$ is DGMS-formal *does* use the general theory. In other words, while the proof does not use the general theory, the implications do.

The most interesting consequence of the DGMS theorem is the following result. If M is a simply connected compact Kähler manifold, then the homotopy groups $\pi_*(M) \otimes \mathbb{R}$ can be deduced entirely from a minimal model for $\mathrm{H}^*_{\mathrm{dR}}(M, \mathbb{R})$.

Another consequence is the following: every Massey product in $\mathrm{C}^*(M, \mathbb{Q})$ vanishes. Indeed, vanishing of a Massey product can be detected after a field extension of the coefficient field. Since they vanish over $\mathbb{R}$, they vanish over $\mathbb{Q}$. This does not mean that $\mathrm{C}^*(M, \mathbb{Q})$ is formal. As discussed last time, the map exhibiting $\mathrm{C}^*(M, \mathbb{R})$ as formal might only be defined over $\mathbb{R}$, or over a finitely generated extension K of $\mathbb{Q}$. The lack of functoriality for formality means that one cannot "descend" the result from $\mathbb{R}$ to $\mathbb{Q}$.

References

[1] A. K. Bousfield and V. K. A. M. Gugenheim, *On* PL *de Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **8** (1976), no. 179, ix+94.

[2] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), no. 3, 245–274.