MATH 215 – Midterm

We will assume the existence of a set $\mathbb{Z}$, whose elements are called integers, along with a well-defined binary operation $+$ on $\mathbb{Z}$ (called addition), a second well-defined binary operation $\cdot$ on $\mathbb{Z}$ (called multiplication), and a relation $<$ on $\mathbb{Z}$ (called less than), and that the following fourteen statements involving $\mathbb{Z}$, $+$, $\cdot$, and $<$ are true:

A1. For all a, b, c in $\mathbb{Z}$, $(a + b) + c = a + (b + c)$.
A2. There exists a unique integer 0 in $\mathbb{Z}$ such that $a + 0 = 0 + a = a$ for every integer a.
A3. For every a in $\mathbb{Z}$, there exists a unique integer $-a$ in $\mathbb{Z}$ such that $a + (-a) = (-a) + a = 0$.
A4. For all a, b in $\mathbb{Z}$, $a + b = b + a$.
M1. For all a, b, c in $\mathbb{Z}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
M2. There exists a unique integer 1 in $\mathbb{Z}$ such that $a \cdot 1 = 1 \cdot a = a$ for all a in $\mathbb{Z}$.
M4. For all a, b in $\mathbb{Z}$, $a \cdot b = b \cdot a$.
D1. For all a, b, c in $\mathbb{Z}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.
NT1. $1 \neq 0$.
O1. For all a in $\mathbb{Z}$, exactly one of the following statements is true: $0 < a$, $a = 0$, $0 < -a$.
O2. For all a, b in $\mathbb{Z}$, if $0 < a$ and $0 < b$, then $0 < a + b$.
O3. For all a, b in $\mathbb{Z}$, if $0 < a$ and $0 < b$, then $0 < a \cdot b$.
O4. For all a, b in $\mathbb{Z}$, $a < b$ if and only if $0 < b + (-a)$.

Notation 1. *We will use the common notation ab to denote $a \cdot b$.*

Notation 2. *We will also use the notation $a > b$ (greater than) to denote $b < a$ (less than).*

We will also assume Propositions 3 through 9. **You do not need to prove these!**

Proposition 3. *For every a in $\mathbb{Z}$, $a \cdot 0 = 0$.*

Proposition 4. *Let a, b be integers. If $ab = 0$, then $a = 0$ or $b = 0$.*

Proposition 5. *0 has no multiplicative inverse. In other words, there is no integer a such that $a \cdot 0 = 1$.*

Proposition 6. *For all a, b, c in $\mathbb{Z}$, if $a + b = a + c$, then $b = c$.*

Proposition 7. *For every a in $\mathbb{Z}$, $-(-a) = a$.*

Proposition 8. *For all integers a and b, $(-a)b = -(ab)$.*

Proposition 9. *For all integer a and b, $(-a)(-b) = ab$.*

The exam is to prove Propositions 10, 11, and 12 on the following pages. You **MAY** use Propositions 1 through 9 in your proofs.

Proposition 10. *For all* a, b *in* $\mathbb{Z}$, $(-a) + (-b) = -(a + b)$.

Proof. By **A3**, $(a + (-a)) + (b + (-b)) = 0 = (a + b) + (-(a + b))$. Thus, $(a + b) + (-(a+b)) = (a+b)+((-a)+(-b))$ using **A1** and **A4**. By Proposition 6, it follows that $-(a + b) = (-a) + (-b)$, as desired. □

Proposition 11. *For all* a *in* $\mathbb{Z}$, $0 < a$ *if and only if* $-a < 0$.

Proof. By **O4**, $-a < 0$ if and only $0 < 0+(-(-a)) = -(-a) = a$, where the first equality is by **A2** and the second is by Proposition 7. □

Proposition 12. *For all* a, b, c *in* $\mathbb{Z}$, *if* $a < b$, *then* $a + c < b + c$.

Proof. If $a < b$, then, by **O4**, $0 < b + (-a) = b + (-a) + 0 = (b + (-a)) + (c + (-c)) = (b + c) + ((-a) + (-c)) = (b + c) + (-(a + c))$, where the first equality follows from **A2**, the second follows from **A3**, the third follows from **A1** and **A4**, and the fourth follows from Proposition 10. Therefore, $a + c < b + c$, again by **O4**. □

Proposition 13. *For all* a, b, c *in* $\mathbb{Z}$, *if* $a < b$ *and* $0 < c$, *then* $ac < bc$.

Proof. If $a < b$, then $0 < b+(-a)$ by **O4**. Hence, since $0 < c$, **O3** says that $0 < (b+(-a))c$. But, $(b + (-a))c = bc + (-a)c = bc + (-(ac))$, where the first equality is by **D1** and the second is by Proposition 8. So, $0 < bc + (-(ac))$, and hence $ac < bc$ by **O4**. □