547 - Spring 2018 - HW5

February 19, 2018

1. Suppose that $0 \to A_* \to B_* \to C_* \to 0$ is a short exact sequence of chain complexes in an abelian category. Prove that if two of the three chain complexes is exact (has zero homology), then so does the third.

2. Prove the 5-lemma. Namely, show that if

$$\begin{array}{ccccccccc} A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle e}\downarrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \end{array}$$

is a commutative diagram in abelian groups with a, b, d, e isomorphisms, then c is an isomorphism.

3. Prove that a submodule of a free abelian group is free.

4. Prove that every projective abelian group is free.

5. For any two abelian groups M and N, compute $\mathrm{Tor}_n^{\mathbb{Z}}(M, N)$ and $\mathrm{Ext}^n_{\mathbb{Z}}(M, N)$ for $n \geqslant 2$.

6. For an abelian group M, let $M[p] = \{x \in M : px = 0\} \subseteq M$. Then, the assignment $M \mapsto M[p]$ defines a left exact functor from abelian groups to itself. Compute the right derived functors.

7. Find $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/m, \mathbb{Z}/n)$ for any i, m, n.

8. Compute $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

9. Compute $\mathrm{Tor}_i^{k[\epsilon]/(\epsilon^2)}(k, k)$ for all $i \geqslant 0$.

10. Say that a chain complex is formal if it is quasi-isomorphic to its homology. Prove that every chain complex over a field is formal.

11. Prove the simplicial identities as shown at `https://ncatlab.org/nlab/show/simplicial+identities`.