548 - Spring 2018 - HW3

February 2, 2018

1. Prove that in $\mathrm{Ch}_{\geqslant 0}(A)$ every morphism $X \xrightarrow{f} Z$ factors as $X \xrightarrow{i} Y \xrightarrow{p} Z$ where $i \in C \cap W$ and $p \in F$. (Every morphism factors as an acyclic cofibration followed by a fibration.) This completes the proof that $\mathrm{Ch}_{\geqslant 0}$ is a model category.

2. Let $\mathcal{C}$ be a model category. Prove that cofibrations are closed under cobase change. In other words, prove that if $X \xrightarrow{i} Y$ is a cofibration and $X \xrightarrow{f} Z$ is any map, then the pushout $Z \xrightarrow{j} Z \cup_X Y$ is a cofibration.

3. Prove that if $\mathcal{C}$ is a model category (with respect to some classes W, C, F), then $\mathcal{C}^{\mathrm{op}}$ admits a natural model category structure.

4. Suppose that $\mathcal{C}$ is a model category. Find a model category structure on $\mathcal{C}$ where the class W of weak equivalences is the class of isomorphisms in $\mathcal{C}$.

5. Based on the work in class with $\mathrm{Ch}_{\geqslant 0}(A)$, propose a model category structure on $\mathrm{Ch}^{\geqslant 0}$, the category of non-negatively graded *cochain* complexes of left A-modules.

6. Prove that if $\mathcal{C}$ is a model category (with respect to W, C, F), then for any object X of $\mathcal{C}$, $\mathcal{C}_{X/}$ admits a model category structure where a map

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ Y & \xrightarrow{\quad f \quad} & Z \end{array}$$

in $\mathcal{C}_{X/}$ is a weak equivalence, cofibration, or cofibration if and only if f is a weak equivalence, cofibration, or fibration in $\mathcal{C}$.

Definition 0.1. Let $\mathcal{C}$ be a model category and let $X \in \mathcal{C}$ be an object. A **cylinder object** for X is an object $X \wedge I$ of $\mathcal{C}$ (this is just a formal symbol) together with maps $X \coprod X \to X \wedge I$ and $X \wedge I \xrightarrow{\sim} X$ whose composition is the fold map $X \coprod X \to X$ induced by $X \xrightarrow{\mathrm{id}_X} X$ and $X \xrightarrow{\mathrm{id}_X} X$. A cyclinder object is **good** if $X \coprod X \to X \wedge I$ is a cofibration and **very good** if additionally $X \wedge I \xrightarrow{\sim} X$ is a fibration, which is necessarily acyclic.

7. Prove that very good cylinder objects exist for every object X of a model category $\mathcal{C}$.