18F568 - Tutorial on infinity-categories

24 September to 5 October 2018

The theory of ∞-categories is one of the main developments of mathematics of the last 15 years. It provides a new packaging of old homotopical ideas that had largely been formulated previously with model categories. The present period is seeing a massive expansion of the use of ∞-categories, both as they reformulate and simplify the proofs of past theorems and as they are used to prove new results that would be more or less inaccessible using the older methods.

The goal of this two week, six lecture series on ∞-categories is to explain the ideas behind the Joyal–Lurie approach using quasicategories, which are certain simplicial sets. These provide a usable, combinatorial model for ∞-categories which has been developed extensively in Lurie's three books [Lur09, Lur17, Lur18] and in the work of many, many other authors.

We will focus on the ideas but not the proofs, which are often rather difficult and involved. Good general sources for introductory material on ∞-categories, besides [Lur09], include [Cis18, Gro15, Rez].

Talk 1 (Ben). The idea of ∞-categories, including simplicial sets, categories, groupoids, Kan complexes, quasicategories, and ∞-categories.

Talk 2. Simplicial categories and the nerve and mapping spaces. This is how one often 'boots up' into the world of ∞-categories. See [Lur09, 1.1.4, 1.1.5, 1.2.2].

Talk 3. The homotopy category. Give both constructions from [Lur09, 1.2.3]. See also [Cis18, Theorem 1.6.6].

Talk 4. Limits and colimits in ∞-categories. Explain products and pullbacks as in [Lur09, 1.2.13]. Explain the general case using [Lur09, 1.2.13.3, 1.2.13.6, 1.2.16.1, 4.2.4.1]. This explains how the obtain the theory of ∞-categorical limits and colimits using just the notion of (∞-categorical) limits and colimits in the ∞-category $\mathcal{S}$ of spaces, which in turn agrees with the notion of homotopy limit and homotopy colimit in the simplicial category (or model category) of topological spaces or simplicial sets (with the Quillen model category structure).

Talk 5. Presheaf categories, Yoneda, accessibility, and presentability. Chapter 5 of [Lur09]. Say something about the proof of the adjoint functor theorem using [Lur09, 5.5.2.7]. The monograph [AR94] of Adámek and Rosicky is a useful introduction to presentability for ordinary categories (where they are called *locally* presentable).

Talk 6 (Ben). A return to direct sum K-theory via ∞-categories (after Gepner–Groth–Nikolaus [GGN15]).

References

[AR94] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994.

[Cis18] D.-C. Cisinski, *Higher categories and homotopical algebra* (2018), available at http://www.mathematik.uni-regensburg.de/cisinski/CatLR.pdf. To appear in Cambridge studies in advanced mathematics.

[GGN15] D. Gepner, M. Groth, and T. Nikolaus, *Universality of multiplicative infinite loop space machines*, Algebr. Geom. Topol. **15** (2015), no. 6, 3107–3153.

[Gro15] M. Groth, *A short course on ∞-categories*, ArXiv e-prints (2015), available at https://arxiv.org/abs/1007.2925.

[Lur09] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Lur17] ———, *Higher algebra* (2017), available at http://www.math.harvard.edu/~lurie/. Version dated 18 September 2017.

[Lur18] ———, *Spectral algebraic geometry* (2018), available at http://www.math.harvard.edu/~lurie/. Version dated 3 February 2018.

[Rez] C. Rezk, *Stuff about quasicategories*, available at https://faculty.math.illinois.edu/~rezk/595-fal16/quasicats.pdf.