# 18F568 - Topics in algebraic topology: higher algebraic $K$-theory

## August 26, 2018

The goal of this course is to introduce higher algebraic $K$-theory, both in the now-classical Waldhausen setting and in the modern $\infty$-categorical setting. The first four weeks will focus on Waldhausen's $K$-theory developed in [Wal85]; we will largely follow [Wei13]. The next two weeks will be a tutorial on $\infty$-categories, as developed in [Lur09]. Other sources might include [Cis18, Gro15, Rez]. The remaining nine weeks will introduce noncommutative motives, cyclotomic spectra, and give the proof of the following theorem, following Raskin [Ras18].

**Theorem 0.1** (Dundas–Goodwillie–McCarthy [DGM13])**.** *Let $A \to B$ be a surjection of associative rings with nilpotent kernel. Then, the trace map induces an equivalence of fibers*

$$\mathrm{fib}(\mathrm{K}(A) \to \mathrm{K}(B)) \to \mathrm{fib}(\mathrm{TC}(A) \to \mathrm{TC}(B)),$$

*where* $\mathrm{K}$ *is nonconnective algebraic $K$-theory and* $\mathrm{TC}$ *is integral topological cyclic homology.*

**Week 1.**  Group completion $K$-theory, motivated by [GGN15].

**Week 2.**  Waldhausen $K$-theory.

**Week 3.**  Additivity and localization.

**Week 4.**  Algebraic $K$-theory of rings and schemes.

**Week 5.**  $\infty$-categories I.

**Week 6.**  $\infty$-categories II.

**Week 7.**  Stable $\infty$-categories. Additive and localizing invariants after [BGT13].

**Week 8.**  Noncommutative motives after [BGT13].

**Week 9.**  Cyclotomic spectra after [NS17].

**Week 10.**   Topological cyclic homology TC, after [NS17].

**Week 11.**   The trace map $\mathrm{K}(-) \to \mathrm{TC}(-)$.

**Weeks 12-15.** The proof of Theorem 0.1. Time-permitting, I will also cover some complementary material from [CMM18].

# References

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