1 September 2015.

Computations in K-theory.

1. What is K-theory?

Grothendieck: A an associative ring,

$$K_0(A) = \left\{ \begin{array}{l} \text{Free abelian group on f.g.} \\ \text{projective right } A\text{-modules} \end{array} \right\} \Big/ \left(\begin{array}{l} [Q] = [P] + [R] \text{ when } 0 \to P \to Q \to R \to 0 \\ \text{is an exact seq. of f.g. projectives} \end{array} \right).$$

Developed for Riemann-Roch theorems.

$G_0(A) = K_0'(A)$ same thing but with f.p. right A-modules.

Dévissage: R regular noetherian commutative $K_0(A) \cong G_0(A)$.

$K_0(A)$ is the group completion of the monoid of iso. classes of f.g. right A-modules under direct sum.

Similar definitions for schemes. Let $Y \subset X$ be a closed inclusion of regular noetherian schemes, $U = X \setminus$ [redacted]. Then,

$$K_0(Y) \to K_0(X) \to K_0(U)$$

is exact. So, this starts to look like some kind of cohomology theory for schemes, or rings. Higher algebraic K-theory works to extend this to a LES.

Quillen/Segal: $+$-construction and group completion.

$$\Omega^\infty K(\mathbf{A}) = \left(\coprod_P BGL(P) \right)^{gp\text{-completion}}.$$

topological monoid under direct sum.

Also, a much more difficult $+$-construction

$$BGL_\infty(A) \longrightarrow BGL_\infty(A)^+,$$

which is a homology isomorphism. This gives some important computational control.

Blumberg-Gepner-Tabuada. K-theory satisfies a recently-discovered universal property. It is the universal [redacted] localizing invariant of small stable ∞-categories [redacted] such that $K(\mathcal{C}) \simeq K(\mathcal{D})$ for $\mathcal{C}, \mathcal{D}$ Morita equivalent. The localizing aspect is crucial for calculations.

So far, however, the universal nature has not been harnessed for computations.

2. K-theory of finite fields and [redacted] rings of integers.

$$K_0(\mathbb{F}_q) \underset{\text{rank}}{\cong} \mathbb{Z},$$

$$\left. \begin{array}{l} K_{2i}(\mathbb{F}_q) = 0, \\ K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1) \end{array} \right\} i \geq 1.$$

Quillen proves this using a description of $K(\mathbb{F}_q)$ as the fiber of

[redacted]

$$ku \xrightarrow{id - \Psi^q} ku,$$

which relates algebraic K-theory directly to complex K-theory.

Cor. $K(\mathbb{F}_q)_{\mathbb{Q}} \simeq H\mathbb{Q}$.

Quillen. $K_i(\mathcal{O}_K)$ is f.g. for each i, where $\mathcal{O}_K$ is the ring of integers in a # field.

Borel. For $i \geq 1$,

$$K_i(\mathcal{O}_K)_{\mathbb{Q}} \cong \begin{cases} 0 & i \equiv 0 \bmod 4, \\ \mathbb{Q}^{r_1 + r_2} & i \equiv 1 \bmod 4, \\ 0 & i \equiv 2 \bmod 4, \\ \mathbb{Q}^{r_2} & i \equiv 3 \bmod 4, \end{cases}.$$

where r_1 is the number of real embeddings of K, r_2 the number of conjugate pairs of complex embeddings.

Table of $K_i(\mathbb{Z})$.

$K_0(\mathbb{Z}) \cong \mathbb{Z}$, $\quad K_4(\mathbb{Z}) = 0$,

$K_1(\mathbb{Z}) = \mathbb{Z}/2$, $\quad K_5(\mathbb{Z}) \cong \mathbb{Z}$,

$K_2(\mathbb{Z}) = \mathbb{Z}/2$, $\quad K_6(\mathbb{Z}) = 0$,

$K_3(\mathbb{Z}) \cong \mathbb{Z}/48$, $\quad K_7(\mathbb{Z}) \cong \mathbb{Z}/240$,

$K_8(\mathbb{Z}) = ?$

Uses Iwasawa theory, motivic cohomology, Quillen-Lichtenbaum

3. Some conjectures.

Conj. $K_{4i}(\mathbb{Z}) = 0$ for $i \geq 1$.

Known only for $i = 1$!
Equivalent to Vandiver's conjecture about class numbers of maximal real subfields of $\mathbb{Q}(\zeta_p)$, namely that $p \nmid \# \mathrm{Pic}(\mathrm{Re}(\mathbb{Q}_p))$.

Conj (Bass). If R is a f.g. [redacted] regular commutative $\mathbb{Z}$-algebra, then $K_i(R)$ is f.g. for each $i \geq 0$.

Known for $\mathcal{O}_K$, $\mathbb{F}_q(C)$.

Conj (Parshin). If $X/\mathbb{F}_q$ is smooth and projective, then $K_i(X) \otimes \mathbb{Q} = 0$ for $i > 0$.

Known for curves.

Any advance would be major.

4. Recent work

$K(\mathbb{S})$, [redacted] dévissage for ring spectra.

5. Trace methods.

K-theory is hard to compute. As an example, $K_8(\mathbb{Z})$ is unknown. It is a [redacted] finite group with smallest prime factor $> 10^7$. There are three modern tools, all of which assume the K-theory of some simple 'basic' rings.

A. Localization. $Z \subseteq X$ closed, $U = X \setminus Z$ complement.

$$K(X \text{ on } Z) \longrightarrow K(X) \longrightarrow K(U)$$

a fiber sequence, which gives LES on K-groups. Due to Thomason.

B. Motivic cohomology.

$$E_2^{pq} = H^p(X, \mathbb{Z}(-\tfrac{q}{2})) \Rightarrow K_{-p-q}(X),$$

X smooth over a field or d.v.r. This is useful for smooth schemes, but the motivic cohomology groups $H^p(X, \mathbb{Z}(-\frac{q}{2}))$ are not so easier to compute either, and are related to Chow groups.

C. Trace methods. Another way of approaching K-theory is to find some theory it maps to and compute that.

Thm (Goodwillie). Let $R \to S$ be a surjection of commutative rings with nilpotent kernel. Then, the fibers of

$$K(R)_{\mathbb{Q}} \longrightarrow K(S)_{\mathbb{Q}} \quad \text{and} \quad [redacted]\ HC^-(R \otimes \mathbb{Q}) \longrightarrow HC^-(S \otimes \mathbb{Q})$$

are the same. HC^- is negative cyclic homology.

Thm (Dundas–McCarthy). Same, but relates fiber of $K(R)^\wedge_p \to K(S)^\wedge_p$ to fiber of $TC(R,p)^\wedge \to TC(S,p)^\wedge$, when TC is topological cyclic homology.