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An overview of computations in algebraic K-theory via trace methods I.

## Algebraic K-theory.

$A$ a ring.

$P(A)$ iso classes of f.g. projective (right) $A$-modules.

Def. Grothendieck $K_0(A)$ is gp completion of $P(A)$. Formally adjoin inverses.

Ex. $K_0(\text{field}) \cong \mathbb{Z}$.

Def (Whitehead). $K_1(A) = GL(A)^{ab} = GL(A)/E(A)$. commutator subgroups.

Milnor defined $K_2(A)$ in the 60s.

Exact sequence $K_2(A,I) \to K_2(A) \to K_2(A/I) \to K_1(A,I) \to K_1(A) \to \cdots \to K_0(A/I)$.

Q. Can we define higher algebraic K-groups extending these with similar exact sequences.

Quillen. $K_n(A) \cong \pi_n(BGL(A)^+)$.

Thm (Quillen). $K_n(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z} & n=0, \\ \mathbb{Z}/q^i-1 & n=2i-1, \\ 0 & \text{otherwise}. \end{cases}$

Next question: what is $K(\mathbb{Z}/p^k)$? Largely unknown. Even if $k=2$.

What about $K(\mathbb{Z})$? Not completely known. Very hard.

Why bother? K-theory touches so many things. Two brief exs.

1) s-cobordism theorem (Mazur, Stallings, Barden). Let $n \geq 5$. Let $W$ be an $(n+1)$-dim cobordism between $n$-manifolds $X, Y$ s.t. $X \hookrightarrow W$ and $Y \hookrightarrow W$ are homotopy equivalences. When is $W \cong X \times [0,1]$? Obstruction lies in $K_1(\mathbb{Z}[\pi_1 X])$. K-theory of group rings.

2) Vandiver's conjecture (Kurihara). $p$ prime, $K$ the maximal real subfield of $\mathbb{Q}(\mu_p)$. Then, $p \nmid$ order of $Cl(K)$. Equivalent to vanishing of $K_{4i}(\mathbb{Z})$, $i>0$.

Q. How do you compute $K_q(A)$?

1960s-70s. Low dimensional using algebra.

Ex. $\mathbb{Z}[x]/(x^n)$.

Thm (Geller-Roberts 1979). $K_2(\mathbb{Z}[x]/(x^2), (x)) \cong \mathbb{Z}/2$.

Trace methods approach. Approximate K-theory by more computable invariants, successive approximations.

Hochschild homology. Simplicial ab. group.

$r \longmapsto A^{\otimes r+1}$

Face maps $\quad d_i(a_0 \otimes a_1 \otimes \cdots \otimes a_r) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_r \\ a_r a_0 \otimes a_1 \otimes \cdots \end{cases}$

$s_i(a_0 \otimes \cdots \otimes a_r) = a_0 \otimes \cdots \otimes 1 \otimes a_{i+1} \otimes \cdots$

$HH_i(A) \cong \pi_i(|HH.(A)|)$.

$HH(A)$ has a cyclic operator given by the cyclic permutation of $a_0 \otimes \cdots \otimes a_r$.

Get a cyclic object; geometric realization has an action of $S^1$.

Dold-Kan. $HH_i(A) \cong H_i(C.(A))$.

$C_n(A) = A^{\otimes n+1}$

$\partial = \sum (-1)^i d_i$.

There is a Dennis trace $K_q(A) \longrightarrow HH_q(A)$.

Use $N.GL_n(A) \to HH(M_n(A)) \xrightarrow{tr} HH(A)$

Goodwillie 80s

Dennis trace factors through $HC^-$:

$$K(A) \longrightarrow HC^-(A) \longrightarrow HH(A).$$

Rationally, $HC^-(A)$ is a good approximation to K-theory.

Double complex

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
A\otimes A\otimes A \arrow[d, "b"] & A\otimes A\otimes A \arrow[l, "1-t"'] \arrow[d, "b'"] & A\otimes A\otimes A \arrow[l, "N"'] \arrow[d, "b"] & \cdots \arrow[l, "1-t"'] \\
A\otimes A \arrow[d, "b"] & A\otimes A \arrow[l, "1-t"'] \arrow[d] & A\otimes A \arrow[l, "N"'] \arrow[d, "b"] & \cdots \arrow[l, "1-t"'] \\
A & A \arrow[l, "1-t"'] & A \arrow[l, "N"'] & \cdots \arrow[l, "1-t"']
\end{tikzcd}$$

$C_n$-norm maps

Products versus sums in the total complexes?

$$b' = \sum_{i=0}^{n-1} (-1)^i d_i.$$

$$b = \sum_{i=0}^{n} (-1)^i d_i.$$

Homology of total complex $HC$.

periodic $HP$.

on the left $HC^-$.

Often written $HN$.

$K(A,I)$ homotopy fiber of $K(A) \to K(A/I)$.
Relative algebraic K-theory.

<u>Thm</u> (Goodwillie). If $I \subset A$ is a nilpotent ideal,

$$K_*(A,I)_{\mathbb{Q}} \simeq HC^-_*(A\otimes\mathbb{Q}, I\otimes\mathbb{Q}).$$

Ex. Thm [Soulé 1981]. rank $K_q(\mathbb{Z}[x]/(x^2),(x)) = \begin{cases} 1 & q \text{ odd}, \\ 0 & q \text{ even}. \end{cases}$ [redacted]

Thm (Stasheff 1985). $K_q(\mathbb{Z}[x]/(x^m),(x))$ is f.g. of rank $m-1$ if $q$ odd, $0$ $q$ even.

Waldhausen: brave new version.

THH (Bökstedt).

Idea: $A \rightsquigarrow HA$

$\otimes \sim \wedge_{\mathbb{S}}$

THH has an $S^1$-action, in fact genuine. Also cyclotomic.

Topological Dennis trace.

$$K(A) \longrightarrow THH(A).$$

Bökstedt-Hsiang-Madsen 93. Topological cyclic homology.

$$\begin{tikzcd}
 & TC(A,p) \arrow[r] & THH(A)^{C_{p^2}} \arrow[d, "F"] \\
 & & THH(A)^{C_p} \arrow[d, "F"] \\
K(A) \arrow[uur, bend left] \arrow[uurr, bend left] \arrow[urr, bend left] \arrow[rr] & & THH(A)
\end{tikzcd}$$

Restriction $R: THH(A)^{C_p} \longrightarrow THH(A)$

$$TC(A,p) \simeq \operatorname*{holim}_{R,F} THH(A)^{C_{p^n}}.$$

[redacted]

Trace factors through $K(A) \to TC(A,p) \to THH(A)$.

Thm (Dundas-Goodwillie-McCarthy). $I \subset A$ nilpotent ideal.

$$K(A,I) \xrightarrow{\simeq} TC(A,I).$$

<u>Thm</u> (Bökstedt–Hsiang–Madsen). $G$ a discrete group st. $H_i(BG,\mathbb{Z})$ f.g. $\forall i \geq 0$. Then,

$$K(\mathbb{Z}) \wedge BG_+ \longrightarrow K(\mathbb{Z}[G])$$

is rationally injective.

<u>Thm</u> (Bökstedt–Madsen 90s). $p$ odd

$$TC(\mathbb{Z})^\wedge_p \simeq \mathrm{im}J^\wedge_p \times B\,\mathrm{im}J^\wedge_p \times SU^\wedge_p$$

$p=2$ analogue of Rognes.

Let $R$ be a ring, f.g. as a $\mathbb{Z}$-module.

$R_p = R \otimes \mathbb{Z}_p$.

$TC(R)^\wedge_p \simeq TC(R_p)^\wedge_p$. Why?

$$\begin{tikzcd} K(\mathbb{Z})^\wedge_p \arrow[r] \arrow[d, "\text{NOT an } \simeq"'] & K(\mathbb{Z}_p)^\wedge_p \arrow[d, "\wr"', "\text{Why?}"] \\ TC(\mathbb{Z})^\wedge_p \arrow[r, "\sim"] & TC(\mathbb{Z}_p)^\wedge_p . \end{tikzcd}$$

How does $HC^-$ behave wrt rationalization?

<u>Early 2000s</u>. Hesselholt–Madsen.

QL conjectures for finite extensions of <u>odd</u> p-adic fields using trace methods.