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Talk II.

Goal: understand $TC(A)$.

Understand $THH(A)^{C_{p^n}} =: TR^{n+1}(A,p)$.

Understand R, F.

$THH(A)$ is an S^1-spectrum.

X a genuine G-spectrum, G compact Lie.

Built out of spaces $X(V)$, V a G-rep.

(a) (Categorical) fixed points: $H \subset G$, X^H.

$$X^H(V) = (X(V))^H$$

$\uparrow$

G-rep fixed by H.

Not well-behaved. For example: not symmetric monoidal.

(b) Geometric fixed points: $\Phi^H X$.

$\mathcal{F}_H$ = subgroups of G not containing H. (H should be normal.)

$E\mathcal{F}_H$ universal $\mathcal{F}_H$ space.

$$(E\mathcal{F}_H)^K \simeq \begin{cases} * & K \in \mathcal{F}_H \\ \emptyset & K \notin \mathcal{F}_H \end{cases}$$

<u>Note</u>. $G = H = C_p$.

$$\mathcal{F}_{C_p} = \{e\}, \quad E\mathcal{F}_H \simeq EC_p.$$

Isotropy sequence:

More nicely behaved.

$$E\mathcal{F}_{H+} \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{F}_H$$

$$\Phi^H X := (\tilde{E}\mathcal{F}_H \wedge X)^H.$$

$$(E\mathcal{F}_{H+} \wedge X)^H \longrightarrow X^H \longrightarrow \Phi^H X.$$

Def. A cyclotomic spectrum is a genuine S^1-spectrum X together with equivalences $r_n : \rho_n^* \Phi^{C_n} X \xrightarrow{\sim} X$ of S^1-spectra for all n.

$$\rho_n : S^1 \to S^1/C_n$$

These should be compatible in the sense that

$$\begin{tikzcd} \rho_n^* \Phi^{C_n}(\rho_m^* \Phi^{C_m} X) \arrow[r, "\simeq"] \arrow[d, "r_m"] & \rho_{nm}^* \Phi^{C_{nm}} X \arrow[d, "r_{nm}"] \\ \rho_n^* \Phi^{C_n} X \arrow[r, "r_n"] & X \end{tikzcd}$$

commute.

Ex. $THH(A)$ is cyclotomic.

$$F : THH(A)^{C_{p^n}} \to THH(A)^{C_{p^{n-1}}},$$
$$R : THH(A)^{C_{p^n}} \to THH(A)^{C_{p^{n-1}}}.$$

I don't quite understand this.

$$\rho_{p^n}^* THH(A)^{C_{p^n}} \simeq \rho_{p^{n-1}}^* \left(\rho_p^* THH(A)^{C_p}\right)^{C_{p^{n-1}}}$$
$$\downarrow$$
$$\rho_{p^{n-1}}^* (THH(A))^{C_{p^{n-1}}}$$

Map on underlying nonequivariant spectra is restriction.

Interested in $THH(A)^{C_{p^n}}$. Need hC_{p^n}.

(c) Homotopy fixed points. $X^{hG} = F(EG_+, X)^G$.

$EG_+ \to S^0$ gives $X^G \xrightarrow{\Gamma} X^{hG}$.

$$\Gamma : THH(A)^{C_{p^n}} \to THH(A)^{hC_{p^n}}.$$

<u>Fundamental diagram.</u> Smash $ES^1_+ \to S^0 \to \widetilde{ES^1}$ with $THH(A)$.

$$\left(\begin{tikzcd}
ES^1_+ \wedge THH(A) \arrow[r] \arrow[d, "\simeq \text{ Why?}"] & THH(A) \arrow[r] \arrow[d] & \widetilde{ES^1} \wedge THH(A) \arrow[d] \\
ES^1_+ \wedge F(ES^1_+, THH(A)) & F(ES^1_+, THH(A)) \arrow[r] & \widetilde{ES^1} \wedge F(ES^1_+, THH(A))
\end{tikzcd}\right)^{C_{p^n}}$$

Rem: $THH(A)$ is cyclotomic

Adams, Greenlees–May

$$\begin{tikzcd}
THH(A)_{hC_{p^n}} \arrow[r] \arrow[d, "\wr"] & THH(A)^{C_{p^n}} \arrow[r, "R"] \arrow[d, "\Gamma"] & THH(A)^{C_{p^{n-1}}} \simeq \left(\Phi^{C_p} THH(A)\right)^{C_{p^{n-1}}} \arrow[d, "\hat{\Gamma}"] \\
THH(A)_{hC_{p^n}} \arrow[r] & THH(A)^{hC_{p^n}} \arrow[r] & THH(A)^{tC_{p^n}}
\end{tikzcd}$$

$$\left(\left(\widetilde{ES^1} \wedge THH(A)\right)^{C_p}\right)^{C_{p^{n-1}}}$$

$$\left(\widetilde{ES^1} \wedge THH(A)\right)^{C_{p^n}}$$

Idea: use

- ability to compute on bottom row. ← spectral sequences.
- understand $\Gamma, \hat{\Gamma}$.
- induction

to understand $THH(A)^{C_{p^n}}$.

<u>Thm</u> (Tsalidis). Suppose $\hat{\Gamma}_1 : \pi_q(THH(A)) \to \pi_q(THH(A)^{tC_p})$ is an iso for $q \geq q_0$, then

$$\hat{\Gamma}_n : \pi_q(THH(A)^{C_{p^{n-1}}}) \to \pi_q(THH(A)^{tC_{p^n}})$$

for $q \geq q_0$.

Rem. Interesting that S^1-equivariant K-theory comes up.

Thm (Hesselholt–Madsen). For k a perfect field of char p.

$$K_{2i-1}\left(k[x]/(x^a),(x)\right) \cong \mathbb{W}_{ai}(k)/V_a\mathbb{W}_i(k),$$

$$K_{2i}\left(\qquad\right) = 0, \quad i>0.$$

Algebraic K-theory of pointed monoid algebras.

$K(A[x]/(x^a))$ — [H-M] $A=k$ perf. of char p

$K(A[x,y]/(xy))$ — [Angeltveit-Gerhardt-Hesselholt] $A=\mathbb{Z}$

[Hesselholt] A = regular $\mathbb{F}_p$-alg —

[Angeltveit-G] $\mathbb{Z}$

$K(A[x_1,\dots,x_n]/(x_1^{a_1},\dots,x_n^{a_n}))$ [Angeltveit-G-Hill-Lindenstrauss].

$K(\mathbb{Z}[C_2])$ Ongoing work of Angeltveit-G.

Pointed monoid algebras.

$\Pi \in \mathrm{Mon}(\mathrm{Sets}_*)$

$A(\Pi) = A[\Pi]/(\text{basepoint})$.

Ex1. $\Pi_n = \{0,1,\dots,x^{n-1}\}$, $x^n=0$

$A(\Pi_n) \cong A[x]/(x^n)$.

Ex2. $\Pi_{x,y} = \{0,1,x,x^2,\dots,y,y^2,\dots\}$ $xy=0$

$A(\Pi_{x,y}) = A[x,y]/(xy)$.

Ex3. $\Pi_{C_2} = \{0,1,x\}$, $x^2=1$ $\mathbb{Z}(\Pi_{C_2}) \cong \mathbb{Z}[C_2]$.

How to approach $K(A(\pi))$.

$THH(A(\pi))$

Fact: $THH(A(\pi)) \simeq THH(A) \wedge N^{cy}(\pi)$

$$[S^q \wedge S^1/C_{p^n +}, THH(A) \wedge N^{cy}(\pi)]_{S^1}.$$

Need S^1-equivariant homotopy type of $N^{cy}(\pi)$.

How to build out of rep. spheres S^λ.

Hard. Potential road block #1.

Suppose that this is OK and we can reduce to

$$[S^q \wedge S^1/C_{p^n +}, THH(A) \wedge S^\lambda]_{S^1}$$
$$\cong$$
$$\pi_{q-\lambda} THH(A)^{C_{p^n}}$$

RO(S^1)-graded homotopy [redacted] [redacted] group.

Potential road block #2.

Not so bad.

First case. $\underline{A = \mathbb{F}_p}$.

Hesselholt-Madsen, G, A-G.

Completely understood.

$\underline{A = \mathbb{Z}}$. Some things understood.

Enough to get some K-theory results.