Hesselholt.

THH and arithmetic: smooth algebras.

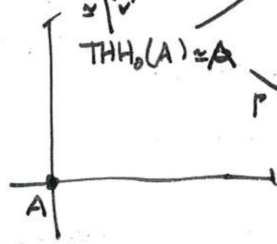
$$
\begin{tikzcd}
THH(A) \arrow[r, "V"] \arrow[d] & TR^2(A,p) \arrow[r] \arrow[d, phantom, "\wr"] & TR^1(A,p) \arrow[d, phantom, "\wr"] & & \\
THH(A)_{hC_p} \arrow[r] \arrow[d] & THH(A)^{C_p} \arrow[r, "R"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & THH(A) \arrow[d, "\phi_p"] & \mathbb{T} \arrow[d] & z \arrow[d, mapsto] \\
THH(A)_{hC_p} \arrow[r] & THH(A)^{hC_p} \arrow[r] \arrow[d, "F"] & THH(A)^{tC_p} & \mathbb{T}/C_p & z^{1/p} \\
& THH(A) & & &
\end{tikzcd}
$$

Now: assume A is $\mathbb{E}_\infty$.

Actually, I think A is just a comm. ring here.

gives edge homomorphism in s.s.

$$
\begin{tikzcd}
TR^1_1(A,p) \arrow[r, "\partial"] \arrow[d, phantom, "\wr"] & \pi_0(THH(A)_{hC_p}) \arrow[r] & TR^2_0(A,p) \arrow[r, "R"] \arrow[dd, "F = \text{th norm. multiplication by } p"] & TR^1_0(A,p) \arrow[d, phantom, "\wr"] \\
HH_1(A,\mathbb{Z}) \arrow[d, phantom, "\wr"] & THH_0(A) \cong A \arrow[u, "\cong"'] \arrow[ur, "V"] \arrow[dr, "p"] & & A \\
\Omega^1_{A/\mathbb{Z}} & & THH_0(A) &
\end{tikzcd}
$$



A

$$E^2_{ij} = H_i(BC_p, THH_j(A)) \Rightarrow \pi_{i+j}\, THH(A)_{hC_p}.$$

Trivial action of C_p on $THH_j(A)$.

Rem. Ω^1 preserves surjections.

Choose $\tilde{A} \to A$ so that $\tilde{A}$ is p-torsion free.

Show that $\partial = 0$ for $\tilde{A}$ and use surjectivity on TR^1_1 to get that $\partial = 0$ for A.

Thus,

$$0 \to A \xrightarrow{V} TR^2_0(A,p) \xrightarrow{R} A \to 0.$$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\Delta_p"] & THH(A) \arrow[d, dashed, "\phi_p"] \\ (A^{\otimes p})^{tC_p} \arrow[r] & THH(A)^{tC_p} \end{tikzcd}$$

Using that $THH(A)$ is universal for circle action, $A^{\otimes p}$ universal for C_p-action.

Only on spaces

$$\begin{tikzcd} & \Omega^\infty A \arrow[dl, dashed] \arrow[d] \\ \Omega^\infty (A^{\otimes p})^{hC_p} \arrow[r] \arrow[d] & \Omega^\infty (A^{\otimes p})^{tC_p} \arrow[d] \\ \Omega^\infty \cancel{THH(A)} THH(A)^{hC_p} \arrow[r] & \Omega^\infty THH(A)^{tC_p} \end{tikzcd}$$

Get $A \to THH(A)^{hC_p}$ and have a section $[-]_2 : A \to TR^2_0(A,p)$.

It is multiplicative but not additive.

$$0 \to A \xrightarrow{V} TR^2_0(A,p) \underset{R}{\overset{[-]_2}{\rightleftarrows}} A \to 0$$

Teichmüller section.

$$A \times A \xrightarrow[([-]_2, V)]{\cong} TR^2_0(A,p) \xrightarrow{(R,F)} A \times A$$

bijection

$$(a_0, a_1) \xmapsto{R} a_0$$

$$(a_0, a_1) \xmapsto{F} a_0^p + p a_1$$

$$(a_0, a_1) \longmapsto (a_0, a_0^p + p a_1).$$

Ring homomorphism

Thm. In this way, there is a canonical ring iso

$$W_n(A) \xrightarrow{\sim} TR^n_0(A,p).$$

Compatible with R, F, V.

Smooth algebras.

de Rham–Witt complex $\xrightarrow{}$ $A\Omega$.
gens + rels $\qquad$ BMS

$$A[x] \simeq A \otimes_{\mathbb{S}} \mathbb{S}[x].$$

$$\mathrm{THH}(A[x]) \simeq \mathrm{THH}(A) \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}[x]).$$

$$\mathrm{THH}(\Sigma^\infty_+ \mathbb{N}) \overset{?}{\simeq} \text{[redacted]}\ \Sigma^\infty_+ \mathcal{L}BN.$$

Exercise. $\mathrm{THH}_*(\mathbb{S}[x]) \simeq \pi_* \mathbb{S} \otimes_{\mathbb{Z}} \Omega^*_{\mathbb{Z}[x]/\mathbb{Z}}$.

$$\mathrm{THH}_*(A[x]) \simeq \mathrm{THH}(A) \otimes^L_{\pi_* \mathbb{S}} \mathrm{THH}_*(\mathbb{S}[x])$$

$$\underset{\text{flatness}}{\simeq} \mathrm{THH}_*(A) \otimes_{\mathbb{Z}} \Omega^*_{\mathbb{Z}[x]/\mathbb{Z}}.$$

I think that we get

$$\mathrm{THH}(\mathbb{F}_p[x]) \simeq \mathrm{THH}(\mathbb{F}_p) \otimes_{\mathbb{S}} \mathrm{THH}(\mathbb{S}[x])$$

$\simeq$ [redacted]

$$\simeq \mathrm{THH}_*(\mathbb{F}_p) \otimes_{\mathbb{F}_p} \Omega^*_{\mathbb{F}_p[x]}$$

$X/\mathbb{F}_p$ smooth,

$\mathrm{THH}(X) \simeq \Omega^*_{X/\mathbb{F}_p}[x]$, $|x| =$

Now, assume that A is a commutative $\mathbb{Z}_{(p)}$-algebra.

$$\bigoplus_{j \geq 0} \mathrm{TR}^2_i(A, p) \xrightarrow{e} \mathrm{TR}^2_i(A[x], p)$$

$$\oplus \bigoplus_{j > 0} \mathrm{TR}^2_{i-1}(A, p)$$

$$\oplus \bigoplus_{j \in \mathbb{Z}_{>0} \setminus p\mathbb{Z}_{>0}} \mathrm{TR}^1_i(A, p)$$

$\eta : A \to A[x]$

(I) $e(0, j, 0)(\omega) = \eta(\omega) \cdot [x]_2^j$

(II) $e(0, j, 1)(\omega) = \eta(\omega) \cdot [x]_2^{j-1} \cdot d[x]_2$ $\qquad$ d is Connes's operator.

$[x]_1 = x$.

(III) $e(1, j, 0)(\omega) = V(\eta(\omega) \cdot [x]_1^j)$

$$V(a \cdot [x]_1^{pj}) = V(a \cdot F([x]_2^j))$$

$$= V(a) \cdot [x]_2^j.$$

Exercise. Show that e is an isomorphism.

(IV) $e(1, j, 1)(\omega) = dV(\eta(\omega) \cdot [x]_1^j)$

k = field

$K_*^{\sqcap}(k) \longrightarrow K_*(k)$

universal example.

$k^\times \longrightarrow K_1(k)$

$l(a)\,l(b) = 0$

$a+b=1$

$W_n\Omega_A^* \longrightarrow TR_*^n(A,p)$

universal example

de Rham-Witt cplx

$W_n(A) \longrightarrow TR_0^n(A,p)$

F, d, V + rels.

Similarly,

$$\bigoplus_{j\geq 0} W_2\Omega_A^i \oplus \bigoplus_{j>0} W_2\Omega_A^{i-1} \oplus \bigoplus_{j\in \mathbb{Z}_{>0}\setminus p\mathbb{Z}_{>0}} \left(W_1\Omega_A^i \oplus W_1\Omega_A^{i-1}\right) \xrightarrow{e} W_2\Omega_{A[x]}$$

Thm. e is a iso. For all comm. $\mathbb{Z}_{(p)}$-algebras A.

H-M. done p odd.

Costeanu $p=2$.

Thm (H.). If A is a smooth $\mathbb{F}_p$-algebra, then the canonical map

TR version.

$$W_n\Omega_A^* \otimes_{W_n(\mathbb{F}_p)} TR_*^n(\mathbb{F}_p,p) \simeq TR_*^n(A,p).$$

proof. (1) By induction for $d\geq 0$, show it is true for $A=\mathbb{F}_p[x_1,\ldots,x_d]$. Use previous thm for induction step.

(2) Étale extensions. If $f: A\to B$ is an étale morphism, then

$$W_n(B)\otimes_{W_n(A)} TR_*^n(A,p) \xrightarrow{\sim} TR_*^n(B,p), \text{ and}$$

$$W_n(B)\otimes_{W_n(A)} W_n\Omega_A^* \xrightarrow{\sim} W_n\Omega_B^*.$$

Thm (Borger / van der Kallen). $A \to B$ étale

$\Rightarrow$ $W_n(A) \to W_n(B)$ is étale.

Also

$$\begin{tikzcd} W_n(A) \arrow[r, "R, F"] \arrow[d] & W_{n-1}(A) \arrow[d] \\ W_n(B) \arrow[r, "R, F"] & W_{n-1}(B) \end{tikzcd}$$

are cocartesian.

$$\begin{tikzcd} W_2(B) \otimes_{W_2(A)} (\pi_* \mathrm{THH}(A)_{hC_p}) \arrow[r] \arrow[d] & W_2(B) \otimes_{W_2(A)} TR^2_*(A,p) \arrow[r] \arrow[d] & B \otimes_A TR^1_*(A,p) \arrow[d, "\cong"', "\text{Geller-Weibel}"] \\ \pi_*(\mathrm{THH}(B)) \arrow[r] & TR^2_*(B,p) \arrow[r] & TR^1_*(B,p) \end{tikzcd}$$

$\cong$ induction

$\cong$ by descent s.s.

and $W_2(A)$-module structure.