Thomas Nikolaus.

On topological cyclic homology and cyclotomic spectra: THH and Frobenius.

$\mathcal{C}$ sym mon ∞-cat.

$(D(\mathbb{Z}), \otimes_{\mathbb{Z}})$, $(Sp, \otimes_{\mathbb{S}})$

$A \in Alg(\mathcal{C})$.

"Def" $HH(A/\mathcal{C}) = \operatorname{colim}_{\Delta^{op}} \left(\cdots A\otimes A\otimes A \substack{\to\\ \to\\ \to} A\otimes A \rightrightarrows A \right)$.

But what about coherences?

$Ass^{\otimes}_{act}$ objects are finite sets.

morphisms are maps of finite sets

equipped with linear order on preimages.

Lexicographic ordering for composition.

Def. An algebra A is a symmetric monoidal functor $N(Ass^{\otimes}_{act}) \to \mathcal{C}$.

Exercise. If $\mathcal{C}$ is a 1-category, check this is $\simeq$ to the ordinary def. Do this.

Now, need $\Delta^{op} \to Ass^{\otimes}_{active}$.

$$S \longmapsto Cut(S) = \{ S_0 \amalg S_1 = S : S_0 < S_1 \} / \emptyset \amalg S = S \amalg \emptyset.$$

Cyclically ordered. So

$$\begin{tikzcd} & \Lambda^{op} \arrow[dr] & \\ \Delta^{op} \arrow[ur, "j"] \arrow[rr] & & Ass^{\otimes}_{act} \end{tikzcd}$$

Now, $\Delta^{op} \to Ass^{\otimes}_{act} \to \mathcal{C}$ gives the cyclic bar construction. So, take colim. This gives $HH(A/\mathcal{C})$. This is the colimit over Δ^{op} of a cyclic set.

$Mod_R \longleftarrow CAlg_R$ preserves sifted colimits, so for A commutative, can compute $HH^R(A)$ in $CAlg_R$.

Why not use $A \otimes_{A^{op} \otimes A} A$? S^1-action not clear.

Cor 1. $\mathcal{C} = D(\mathbb{Z})$

$\mathrm{Alg}(\mathbb{Z}) \leftarrow \text{- - - -}$ DGA A, flat

$\mathrm{Ass}^{\otimes}_{\mathrm{act}} \longrightarrow D(\mathbb{Z})$ ↑ needed to get that this is

$S \longmapsto A^{\otimes S}$. actually symmetric monoidal.

In particular, we recover the classical HH.

Non-flat you get Shukla homology.

Ex. R ordinary commutative ring, flat over $\mathbb{Z}$.

$HH_0(R) = R$.

$HH_1(R) \simeq \Omega^1_{R/\mathbb{Z}}$.

(If R is commutative, then $HH_*(R)$ is graded commutative.)

If R is smooth,

$$\Omega^*_{R/\mathbb{Z}} \xrightarrow{\simeq} HH_*(R)$$

Thomas claims: always have a map $\Omega^*_{R/\mathbb{Z}} \longrightarrow HH_*(R)$.

If R is not smooth, a filtration on $HH(R)$ whose gradeds are the derived symmetric powers of the cotangent complex.

Take a resolution

$$|R_\bullet| \simeq R$$

by polynomials. So, need to derive symmetric powers.

Ex. $R = \mathbb{F}_p$. $|x| = 2$

$HH_*(\mathbb{F}_p) \simeq \mathbb{F}_p\langle x\rangle$ = free divided power algebra on $|x| = 2$.

$$\mathbb{F}_p[x^{(1)}, x^{(2)}, x^{(3)}, \dots] / x^{(i)}x^{(j)} = \binom{i+j}{i} x^{(i+j)}$$

$$\simeq \mathbb{F}_p\left[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \dots\right]$$

$$x^p = p!\left(\frac{x^p}{p!}\right) = 0.$$

Fancy proof. $\mathbb{F}_p$ is lci over $\mathbb{Z}$. So get a shift of the cotangent complex to degree 2.

Always: $HH(R) \simeq R \otimes_{R \otimes R^{op}} R$.

$$\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq \mathbb{Z}[\varepsilon]/(\varepsilon^2) \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq \mathbb{F}_p[\varepsilon]/(\varepsilon^2). \quad |\varepsilon| = 1.$$

Resolve $\mathbb{F}_p$ over $A = \mathbb{F}_p[\varepsilon]/(\varepsilon^2)$.

$$\mathbb{F}_p \simeq A\langle x \rangle, \quad dx^{(i)} = \varepsilon x^{(i-1)}.$$

cdga.

So, $\mathbb{F}_p \otimes_A \mathbb{F}_p \simeq A\langle x \rangle \otimes_A \mathbb{F}_p \simeq \mathbb{F}_p\langle x \rangle$.

no differential.

Case 2. $\mathcal{C} = Sp$.

$A \in Alg(Sp)$.

$HH(A/Sp) =: THH(A)$.

$THH(\mathbb{S}) \simeq \mathbb{S}$.

X connected.

$THH(\mathbb{S}[\Omega X]) \simeq \Sigma^\infty_+ \mathcal{L}X \longleftarrow$ Rotation of loops gives the S^1-action.

Goodwillie–Jones.

Prop. If $A \in Alg(D(\mathbb{Z}))$, there is a map

$$THH(HA) \longrightarrow H(HH(A)) \qquad THH(A) \longrightarrow HH(A).$$

and for A connective, the induced map

$$THH_i(A) \to HH_i(A)$$

is an iso for $i \leq 2$. Also, surjection in deg 3. : Lurie.

Thm (Bökstedt).

$THH_*(\mathbb{F}_p) \simeq \mathbb{F}_p[x]$.

Recall: $\Delta^{op} \xrightarrow{j} \Lambda^{op} \xrightarrow{X} \mathcal{C}$.

Prop. For any $\Lambda^{op} \to \mathcal{C}$, there is a natural $\mathbb{T} = U(1) = S^1$-action on $\operatorname{colim}_{\Delta^{op}} j^* X$.

Ex. In $D(\mathbb{Z})$ a $\mathbb{T}$-action is the same as an action of the algebra $H_*(S^1, \mathbb{Z}) \simeq \mathbb{Z}[b]/(b^2)$, $|b| = 1$.
So, just have to give the b-operator.

Gives the de Rham differential on $\Omega^*_{R/\mathbb{Z}} \cong HH_*(R)$.

Def. $A \in \mathrm{Alg}(\mathcal{C})$. $\mathcal{C}$ preadditive.

$HH(A/\mathcal{C})^{hS^1} =: HC^-(A/\mathcal{C})$.

$HH(A/\mathcal{C})^{tS^1} =: HP(A/\mathcal{C})$.

Equivalent to old defs by Marc Hoyois.

If A is commutative, so are $HC^-(A/\mathcal{C})$, $HP(A/\mathcal{C})$!!!

Rem. I guess HC is a non-unital ring.

For $\mathcal{C} = D(\mathbb{Z})$ $\quad HP_*$, HC^-_*.

periodic de Rham cohomology.

Lars: TP_*, TC^-_*.

top. periodic cyclic homology.

Extra structure on $THH(A)$ which does not exist for $HH(A)$ in general.

<u>Claim</u>. There is a canonical map $\hat{\Gamma}_1 = \phi_p : THH(A) \longrightarrow THH(A)^{tC_p}$ which is S^1-equivariant. It is $\mathbb{E}_\infty$ if A is.

No need for equivariant homotopy theory.

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "\Delta_p"'] & THH(A) \simeq S^1 \otimes A \arrow[d, squiggly] \\
(A \otimes \cdots \otimes A)^{tC_p} \arrow[r] & THH(A)^{tC_p}.
\end{tikzcd}
$$

<u>Def</u>. $$TC(A) \simeq \mathrm{Eq}\left(THH(A)^{h\mathbb{T}} \underset{\phi_p^{h\mathbb{T}}}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_p \left(THH(A)^{tC_p}\right)^{h\mathbb{T}/C_p} \right).$$

$$
\begin{tikzcd}
\mathrm{Mod}_R \arrow[r, bend left=20, "\otimes_R S"] & \mathrm{Mod}_S \arrow[l, "U"'] \arrow[l, phantom, bend right=40, "\perp"] \\
\end{tikzcd}
\qquad \mathrm{Map}_R(S,-),\ \perp
$$

$\mathbb{S} \longrightarrow \mathbb{Z}$

$$\mathrm{Map}_R(UM, N) \cong \mathrm{Map}_S(M, \mathrm{Map}_R(S,N))$$

$$\mathrm{Map}_R(UM, N) \cong \mathrm{Map}_S(M, CN)$$

$M = S$

$$\mathrm{Map}_R(S,N) \simeq_R CN$$

$$
\begin{tikzcd}
 & Z \arrow[d, "p"] \\
X \arrow[ur, "j"] \arrow[r, "f"'] & Y
\end{tikzcd}
$$

$f_! \simeq p_* \circ j_!$