Royes.

The Dundas–Goodwillie–McCarthy Theorem II.

Derivatives.

Thm. $R \to S$ $\mathbb{E}_1$, connective, $\pi_0 R \to \pi_0 S$ surjective w/ nilpotent kernel. Then,

$$\begin{array}{ccc} K(R) & \xrightarrow{\text{trc}} & TC(R) \\ \downarrow & & \downarrow \\ K(S) & \xrightarrow{\text{trc}} & TC(S) \end{array}$$

is cartesian.

Last time, reduced to $R \simeq S \oplus M$, [redacted] split square-zero, M connective bimodule.

$$\partial_S \tilde{K} \simeq \partial_S \tilde{TC} \simeq \Sigma^{-1} THH(R,-).$$

Everything [redacted] said for $\partial_S \tilde{K}$ is literally true for R, S, M discrete. Probably true for connective as well.

$\mathcal{C}$ category with hty theory and cofiber seqs.

E.g. Exact [redacted] cats.

E.g. Waldhausen cats.

E.g. Stable ∞-cats.

Def. $X \in S_n\mathcal{C} \subseteq Fun(Ar[n+1], \mathcal{C})$

s.t. 1) $X(i,i) \simeq *$,

2) $i \leq j \leq k$, $X(i,j) \to X(i,k) \to X(j,k)$ a cofiber seq.

$S_0\mathcal{C} \simeq *$,

$S_1\mathcal{C} \simeq \mathcal{C}$,

$S_2\mathcal{C} \simeq Fun(\Delta^1, \mathcal{C})$.

$\vdots$

$S_n\mathcal{C} \simeq Fun(\Delta^{n-1}, \mathcal{C})$.

Fact. $S_\bullet\mathcal{C}$ is a simplicial object.

Def. $\Omega^\infty K(\mathcal{C}) \simeq \Omega |\iota S_\bullet \mathcal{C}|$.

Thm (Waldhausen). $\exists$ natural maps for $n \geq 0$

$$\Omega^n |iS_\bullet^{(n)} \mathcal{C}| \longrightarrow \Omega^{n+1} |iS_\bullet^{(n+1)} \mathcal{C}|.$$

Equivalences for $n \geq 1$.

Get a spectrum $K(\mathcal{C})$.

Motto. Taking K-theory is like $\Omega^\infty \Sigma^\infty$, or like a suspension spectrum functor.

This is made precise in BGT1, Barwick.

R connective $\mathbb{E}_1$-ring, M connective bimodule.

Two most stable ∞-cats associated to (R, M).

1) $\mathrm{Mod}^\omega_{R \oplus M}$.

2) $\mathrm{End}(R, M)$ ∞-cat of M-valued endomorphisms.

Precise definition. $P \xrightarrow{f} P \otimes_R M$, P perfect.

Morphisms

$$\begin{array}{ccc} P & \longrightarrow & P \otimes_R M \\ g \downarrow & & \downarrow g \otimes \mathrm{id}_M \\ Q & \longrightarrow & Q \otimes_R M \end{array}$$

Conjecture. $\mathrm{Mod}^\omega_{R \oplus M} \simeq_{\mathrm{Mot}} \mathrm{End}(R, \Sigma M)$ in some cases.

Guess: not true.

$\longleftarrow$ Inverse is supposed to be

$$(P \to P \otimes_R M) \longmapsto (P \otimes_R M \to P' \to P)$$

and P' has a canonical $R \oplus M$-module structure.

$$P' \wedge (R \oplus M) \simeq (P' \otimes_R R) \vee (P' \otimes_R M)$$
$$\to P' \oplus (P \otimes_R M) \to P' \oplus P' \to P'.$$

Use additivity.

$$K(\mathrm{End}(R,M)) \underset{\text{Claim; true in discrete case.}}{\simeq} \Omega \left| \coprod_{\bar{P} \in S_\cdot \mathrm{Mod}_R^\omega} M \cdot p_{\mathrm{sMod}_R}(\bar{P}, \bar{P} \otimes_R M) \right| .$$

I think this is just true using weight structures.

<u>Conj</u>. $$\mathrm{THH}(R,M) \simeq \operatorname*{colim}_{n \to \infty} \Omega^n \left| \bigoplus_{\bar{P} \in S_\cdot^{(n)} \mathrm{Mod}_R^\omega} M \cdot p(\bar{P}, \bar{P} \otimes_R M) \right|$$

$$\uparrow$$

$$\operatorname*{colim}_{n \to \infty} \Omega^n \left| \bigvee_{\bar{P} \in S_\cdot^{(n)} \mathrm{Mod}_R^\omega} M \cdot p(\bar{P}, \bar{P} \otimes_R M) \right| .$$

As connectivity of $M \to \infty$, this map becomes an $\simeq$.

$$\wr\| \text{ claim}$$

$$\Sigma \tilde{K}(R \oplus M).$$

Turning to TC. Work of Aaron on the Taylor tower of TC.

$\mathrm{TC}(R \oplus M)$, R, M connective.

Following Hesselholt's thesis and Nikolaus-Scholze.

$$\mathrm{THH}(R \oplus M) \simeq |B^{cyc}_\cdot(R \oplus M)|$$

$$B^{cyc}_n(R \oplus M) \simeq (R \oplus M)^{\otimes n+1}$$

$$\simeq \bigvee_{S \subset [n]} R^{\otimes [n] \setminus S} \otimes M^{\otimes S}$$

$$T_{a,n}(R,M) = \bigvee_{\substack{S \subset [n] \\ |S| = a}} R^{\otimes ([n] \setminus S)} \otimes M^{\otimes S}.$$

<u>Prop</u> (Hesselholt).

$T_{a,\cdot}$ is a cyclic subspectrum.

⚠ $|T_{a,\cdot}| = |T_{a,\cdot}(R,M)|$ is not cyclotomic.

Instead, there is a non-trivial component

$$\begin{array}{ccc} THH(R \oplus M) & \xrightarrow{\phi_p} & THH(R\oplus M)^{tC_p} \\ \uparrow & & \uparrow \\ T_a(R,M) & \dashrightarrow & T_{pa}(R,M). \end{array}$$

<u>Prop.</u> $THH(R\oplus M)$ has a filtration by cyclotomic spectra

$$\begin{array}{l} \vdots \\ F_1 = \bigoplus_{a=0}^{1} T_a(R,M) \\ \downarrow \\ F_0 = THH(R) = T_0(R,M) \end{array}$$

Moreover, $F_a/F_{a-1} \simeq T_a(R,M) \simeq THH(R, M^{\otimes a}) \otimes_{C_a} S^1$.

$THH(R,M^{\otimes a}) \otimes_{C_a} S^1 \simeq U^a(R,M)$ in the notation of Lindenstrauss–McCarthy.

$\Rightarrow$ Each F_a/F_{a-1} is cyclotomic.

But, the cyclotomic ϕ_p are zero.

<u>Ex.</u> $F_1 = THH(R,M) \wedge \mathbb{T}_+ \longrightarrow (THH(R,M)\wedge \mathbb{T}_+)^{tC_p} = 0.$

So,

$$TC(F_a/F_{a-1}) \longrightarrow (F_a/F_{a-1})^{h\mathbb{T}} \xrightarrow{\text{can}-\phi_p \;(=0)} \prod_{p \text{ prime}} \left(F_a/F_{a-1}\right)^{tC_p\, h\mathbb{T}/C_p}$$

$$(F_a/F_{a-1})^{h\mathbb{T}} \simeq \left(\left(F_a/F_{a-1}\right)^{hC_a}\right)^{h\mathbb{T}/C_a}$$

So, get $THH(R,M^{\otimes a})_{hC_a}$. Lindenstrauss–McCarthy.

$$\Sigma^{-1}\left(\left(F_a/F_{a-1}\right)_{hC_a}\right)^{h\mathbb{T}/C_a}$$

$$\simeq$$

$$\Sigma^{-1}\left(\left(THH(R,M^{\otimes a})\otimes_{C_a} S^1\right)_{hC_a}\right)^{h\mathbb{T}/C_a}.$$

Recall that TC commutes with limits of cyclotomic spectra.

Assume M is [illegible] n-connected. Then,

$TC(F_a/F_{a-1})$ is an-connected.

Thus,

$$\operatorname{colim} \Sigma^{-n} \widetilde{TC}(R \oplus \Sigma^n M)$$

$$\simeq TC(F_1/F_0)$$

$$\simeq \Sigma^{-1} THH(R, M).$$

<u>Thm.</u> ~~$TC(F_a) \simeq$~~ The F_a filtration is precisely the Goodwillie-Tower of

$$TC(R \oplus -): {}_R\mathrm{Mod}_R \longrightarrow Sp.$$

Generalizes top. Witt vector construction of Lindenstrauss-McCarthy.