VITAMIN K_1: KERZ-STRUNK-TAMME'S SOLUTION TO WEIBEL'S CONJECTURES

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INTRODUCTION

In this workshop, we will cover the proof of Weibel's conjecture.

Theorem 1 (Weibel's conjecture; Kerz-Strunk-Tamme)**.** *Suppose that X is a Noetherian scheme of finite Krull dimension d. Then for $i < -d$ the group $K_i(X)$ vanishes.*

The proof of Weibel's conjecture follows from a "pro-descent" theorem for nonconnective algebraic K-theory. Recall that an abstract blowup square is a Cartesian square

(1)
$$\begin{array}{ccc} E & \longrightarrow & \widetilde{X} \\ \downarrow & & \downarrow p \\ Z & \xrightarrow{i} & X, \end{array}$$

where i is a closed immersion, p is a proper map, and the induced map on complements $\widetilde{X} \setminus E \to X \setminus Z$ is an isomorphism. In particular, blowups are abstract blowup squares. It is known that K-theory does not take an arbitrary abstract blowup square to a Cartesian square of spectra. However, if one takes into account the infinitesimal thickenings of E and Z in $\widetilde{X}$ and X Kerz, Strunk, and Tamme obtain the following result, from which they deduce Theorem 1.

Theorem 2 (Kerz-Strunk-Tamme)**.** *For any abstract blowup square as in (1), we have a Cartesian diagram of pro-spectra*

(2)
$$\begin{array}{ccc} K(X) & \longrightarrow & \text{“}\lim\text{”} K(Z_n) \\ \downarrow & & \downarrow \\ K(\widetilde{X}) & \longrightarrow & \text{“}\lim\text{”} K(E_n) \end{array}$$

where Z_n (resp. E_n) is the nth infinitesimal thickening of Z in X (resp. E in $\widetilde{X}$).

Theorems 1 and 2 are the latest installments in a long series of results exploring vanishing of negative K-theory and cdh-descent (for which, see the references and the introduction to [12]). The goal of this workshop is to understand that history in the large, the details of the proof of Theorem 2, and how this implies Theorem 1.

Day 1: Preliminaries

Talk 1 (Daniel Carmody): Descent and cd-structures: 1 hour. Following [2, Section 2.1] define cd-structures and give examples — the Zariski topology, the Nisnevich topology, and the cdh-topology; explain how they generate a Grothendieck topology. In particular, define elementary distinguished squares that define Nisnevich topology and the abstract blow-up squares that define the cdh-topology [19, Definition 12.21]. Define what it means for a presheaf (of spaces/spectra) on a small category with a cd-structure to be excisive [2, Definition 3.2.1] and explain Voevodsky's theorem that, under certain assumptions, this is equivalent to the presheaf satisfying descent with respect to the topology generated by the cd-structure [2, Theorem 3.2.5]. Explain that excision gives rise to Mayer-Vietoris-style long exact sequences. Other relevant references are [26] and [27].

Talk 2 (Gabe Angelini-Knoll): Pro-objects in ∞-categories: 1 hour. Quickly review the classical definition of pro-objects as formal cofiltered limits [1]. Explain the ∞-categorical formulation following [18, Section A.8.1] or [4], including the computation of mapping spaces in pro-categories and the universal property of pro-objects. Explain the notion of *weak equivalences* of pro-spaces and pro-spectra following [12, Section 4.1]; a similar discussion can be found in [13, Section 2]. Discuss the notion of pro-descent and pro-excision following [17].

Talk 3 (Brian Shin): Negative K-theory: 1.5 hours. Briefly recall the definition and universal property of connective K-theory of a stable ∞-category following [6, Section 7]. Do the same for nonconnective K-theory following [6, Section 9]. Construct the Bass model for nonconnective K-theory of schemes following [25, Section 6] and explain how this relates to the definition in [6]. Explain why for $j > 0$, the group $K_{-j}(Y)$ is a quotient of $K_0(Y \times \mathbf{G}_m^j)$ and the fact that if an element in the latter group comes from $K_0(Y \times \mathbf{A}^j)$, then it vanishes in $K_{-j}(Y)$. Mention that nonconnective K-theory satisfies localization [25, Section 7] and therefore K-theory satisfies Zariski (and Nisnevich) descent.

Talk 4 (Harry Smith): Genesis of Weibel's conjecture: 1.5 hours. Weibel's conjecture appears as (the second part of) Question 2.9 in [29]. Introduce the conjecture. Explain the computations of Bass cited in [29, Proposition 2.8] showing the conjecture is true in dimensions ≤ 1 and Weibel's verification of the conjecture in dimension 2 [30, Theorem 4.4]. Give an overview of some of the previous work on Weibel's conjecture, starting with Haesemeyer's proof that homotopy K-theory satisfies *cdh*-descent in characteristic zero given in [9, Theorem 6.4] or [8, Theorem 3.12]. Time permitting, one might also mention Cisinksi's proof in [7], using 6-functors and motivic homotopy. Say a few words about Cortiñas-Haesemeyer-Schlichting-Weibel's proof [8, Corollary 5.9] that Weibel's conjecture holds in characteristic zero and Kelly's proof [10, Theorem 3.5] that it holds for $K[1/p]$ in characteristic p.

Day 2: Homotopy K-Theory, Excision and Beginnings of the Proof

Talk 5 (Yifei Zhao): The case of homotopy K-theory: 1.5 hours. Explain in detail the proof of Weibel's conjecture for homotopy K-theory due to Kerz and Strunk [11]. First, define homotopy K-theory following [28]. Then state, without proof, the two main ingredients needed for the proof: first is the fact that homotopy K-theory satisfies cdh-descent [7] and the second a theorem of Raynaud and Gruson [20, Theorem 5.22] on "platification par éclatement". Proceed to give the proof [11, Theorem 1] in full. Explain why this gives Weibel's conjecture after appropriate inversion of primes [11, Corollary 2].

Talk 6 (Tasos Moulinos): Suslin–Wodzicki excision after Tamme: 1.5 hours. This is a talk on Tamme's proof [23] of Suslin–Wodzicki excision [22], [21]. The goal is to give a proof of [23, Theorem 21] in full detail. Define the lax pullbacks of ∞-categories following [23, Section 1] Define Milnor squares and give examples. Prove [23, Theorem 11] in detail. Explain the notion of Tor-unitality after Tamme [23, Definition 12] and then prove the main result [23, Theorem 16] and explain how one obtains Suslin–Wodzicki excision for any localizing invariant [23, Section 3].

Talk 7 (Benjamin Antieau): K-Theory of Derived Schemes: 1 hour. Give an overview of derived algebraic geometry in the context of simplicial commutative rings following [12, Section 2.1]; a useful additional reference is [14]. Define the ∞-category of perfect complexes (see [12, Section 2.1] or [5] for more details) of derived schemes so that one can take algebraic K-theory of a derived scheme. Explain that the connective K-theory of affine derived schemes can be computed via the plus construction [12, Proposition 2.15]. Prove the nilinvariance result [12, Theorem 2.16]: the K-theory of an affine derived scheme is equivalent to the K-theory of its underlying scheme, upon taking 1-truncation.

Talk 8 (Elden Elmanto): Derived Blowups: 1.5 hours. Define derived blowups and prove the descent theorem [12, Theorem 3.7] for them. First, as motivation, explain Thomason's classical result on descent for blowups along regularly immersed centers [24]; the main result of [12, Section 3] is analogous to this. Explain the notions of derived blowups, semi-derived and derived exceptional divisors, using the diagram [12, (3.3)] as a guide. Prove that derived blowups are independent of all auxiliary choices [12, Lemma 3.6]. Prove [12, Theorem 3.7] in detail. Explain how to deduce the projective bundle formula [12, Theorem 3.16] and Bass' fundamental theorem [12, Theorem 3.17].

DAY 3: CONCLUDING THE PROOF

Talk 9 (Joel Stapleton): Pro-excision for simplicial rings: 1.5 hours. Give some motivation for pro-excision from Grothendieck's theorem on formal functions [12, Introduction] or defining K-theory with compact supports following [15, Section 4.1]. Explain why K-theory cannot, in general, take abstract blowup squares to Cartesian squares. State clearly the pro-excision result [12, Theorem 4.11] and explain how to deduce [12, Corollary 4.13] which we will need in the proof of the main theorem. Explain how [12, Corollary 4.13] proves a pro-equivalence between relative K-theory of derived and underived schemes. Proceed to prove [12, Theorem 4.11].

Talk 10 (Jeremiah Heller): Proof of Theorem 2, part 1: 1.5 hours. State clearly [12, Theorem A] and proceed with the proof. The first step is to prove [12, Theorem A] for the case that $\tilde{X} \to X$ is a finite morphism. This is [12, Proposition 5.2]. Next, we reduce from the case of arbitrary abstract blowup squares to the case that of a classical blowup. This relies on some ideas that were already discussed in Talk 5, namely, "platification par éclatement." Prove [12, Claim 5.3] in detail.

Talk 11 (Aron Heleodoro): Proof of Theorem 2, part 2: 1 hour. Explain how to reduce to the case of derived blowups, this is [12, Lemma 5.5] after the efforts of Talk 9. Explain the appearance of the tower of derived blowups [12, Section 5.4] and conclude the proof of [12, Theorem A].

Talk 12 (Marc Hoyois): Proof of Theorem 1: 1.5 hours. Prove Weibel's conjecture [12, Theorem B] — this follows the outline for the case of homotopy K-theory and references to material covered in Talk 5 should be made. Prove that homotopy K-theory satisfies cdh-descent (this is [12, Theorem C]) — this reproves Cisinski's theorem [7]. Mention, without proof, that the cdh-sheafification of K-theory is in fact homotopy K-theory [12, Theorem 6.3].

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