Math 331 discussion problems

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These are extra practice problems, not to be handed in.

1. (a) Find the prime factorization of $10 + 91i$ in $\mathbb{Z}[i]$.

   (b) Prove that an element of $\mathbb{Z}[i]$ has even norm if and only if it is divisible by $1 + i$.

2. (a) Let R be a UFD. Let $a, b \in R$ be nonzero and relatively prime. Suppose there exists $c \in R$ and $n \in \mathbb{N}$ such that $ab = c^n$. Prove there exist elements $x, y \in R$ and units $u, v \in R^\times$ such that $a = ux^n$, $b = vy^n$.

   (b) Prove, using the previous part, that the only integer solutions to $y^2 + y = x^3$ are $(0, 0)$ and $(0, -1)$. Conclude that 0 is the only integer which is both a cube and the product of two consecutive integers.

3. For every prime $p \in \mathbb{Z}_{\geq 0}$, let

$$\psi_p : \mathbb{Z}[x] \to (\mathbb{Z}/p\mathbb{Z})[x]$$

   be the ring homomorphism given by "reduction of coefficients",

$$\psi_p(a_n x^n + \cdots + a_0) = \overline{a}_n x^n + \cdots + \overline{a}_0$$

   where $\overline{a} \in \mathbb{Z}/p\mathbb{Z}$ is the image of $a \in \mathbb{Z}$ under the projection map. Let $f \in \mathbb{Z}[x]$ have positive degree. Prove that the gcd of the coefficients of f is 1 if and only if $\psi_p(f) \neq 0$ for all primes p.