Math 331 discussion problems

TA: Alex Karapetyan

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These are extra practice problems, not to be handed in.

1. Let R be a ring. An element $x \in R$ is *nilpotent* if there exists some positive integer n such that $x^n = 0$. Prove that if $x \in R$ is nilpotent then $1 + x$ is a unit.

2. Consider
$$f(x) = 4x^2 + 6x + 3 \in (\mathbb{Z}/8\mathbb{Z})[x]$$
Is f a unit in $(\mathbb{Z}/8\mathbb{Z})[x]$?

3. Let R be a commutative ring and let I, J be ideals in R. Prove that
$$\{r \in R : rJ \subseteq I\}$$
is also an ideal in R.

4. Let I be the principal ideal generated by $x^2 + x + 2$ in the ring $R = (\mathbb{Z}/5\mathbb{Z})[x]$. Find the multiplicative inverse of $2x + 3 + I$ in R/I.

5. Prove that the map
$$\phi : \mathbb{R}[x] \to \mathbb{R}[\epsilon]/(\epsilon^2)$$
given by
$$\phi(p(x)) = p(0) + p'(0)\epsilon$$
is a ring homomorphism.