Solution to Problem 2 in the SIAM Activity Group on Orthogonal Polynomials and Special Functions Newsletter

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Problem 2

Is it true that

$$x^2 t^x {}_2F_1(x+1, x+1; 2; 1-t)$$

is a convex function of x whenever $-\infty < x < \infty$ and $0 < t < 1$?

Submitted by George Gasper, August 19, 1992.

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Let $0 < t < 1, -\infty < x < \infty$, and set

$$f_t(x) = x^2 t^x {}_2F_1(x+1, x+1; 2; 1-t). \tag{1}$$

By using the binomial theorem to expand $t^x = (1-(1-t))^x$ in powers of $1-t$, it follows that

$$f_t(x) = x^2 \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-x)_j (1+x)_k (1+x)_k}{j!\, k!\, (2)_k} (1-t)^{j+k}$$

$$= x^2 \sum_{n=0}^{\infty} \frac{(-x)_n}{n!} (1-t)^n {}_3F_2(-n, 1+x, 1+x; 2, 1+x-n; 1)$$

after setting $j = n-k$ and changing the order of summation. Then, application of the $b = -n$ case of the transformation formula [1, 3.8(1)] to the above ${}_3F_2$ series yields the expansion formula (generating function)

$$f_t(x) = x^2 \sum_{n=0}^{\infty} F_n(x) (1-t)^n \tag{2}$$

with

$$F_n(x) = {}_3F_2(-n, 1+x, 1-x; 2, 1; 1), \quad n = 0, 1, \ldots. \tag{3}$$

From (3) it is clear that each $F_n(x)$ is a polynomial of degree n in x^2, and hence, by (2), that $f_t(x)$ is an even function of x. Also, computations of the coefficients of the polynomials $F_n(x)$ for many values of n suggest that each $F_n(x)$ is an absolutely monotonic function (one whose power series coefficients are nonnegative) of x. Then (2) would imply that $f_t(x)$ is an absolutely monotonic function of x and, consequently, a convex function of x when $0 < t < 1$. Thus it suffices to prove that each $F_n(x)$ is an absolutely monotonic function of x.

First observe that

$$F_n(x) = \frac{1}{n!\,(n+1)!} S_n(-x^2; 1, 1, 0), \tag{4}$$

where

$$\frac{S_n(x^2; a, b, c)}{(a+b)_n (a+c)_n} = {}_3F_2(-n, a+ix, a-ix; a+b, a+c; 1), \tag{5}$$

n=0,1,..., are the continuous dual Hahn polynomials, see [4, §1.3]. Since the polynomials $\{S_n(y; a, b, c)\}_{n \geq 0}$ are orthogonal on the interval $(0, \infty)$ with respect to a positive weight function when $c \geq 0$ and either $a, b > 0$ or a, b are complex conjugates with positive real parts, the zeros $y_{k,n}(a, b, c), 1 \leq k \leq n$, of each $S_n(y; a, b, c)$ are positive, and hence

$$S_n(x^2; a, b, c) = C_n(a, b, c) \prod_{k=1}^{n} \left(y_{k,n}(a, b, c) - x^2 \right)$$

where, by inspection of the right side of (5), $C_n(a, b, c) > 0$. Therefore, with the above-mentioned restrictions on a, b, c each

$$S_n(-x^2; a, b, c) = C_n(a, b, c) \prod_{k=1}^{n} \left(y_{k,n}(a, b, c) + x^2 \right) \tag{6}$$

is an absolutely monotonic function of x, and hence, by (4) and (2), $F_n(x)$ and $f_t(x)$ are absolutely monotonic functions of x, which completes the proof. From this proof it is clear that the function $t^x {}_2F_1(x+1, x+1; 2; 1-t)$ is also an absolutely monotonic, and hence convex, function of x. For a discussion of the origin of this problem, see page 11 of the Fall 1994 issue of this Newsletter (note that in the fourth displayed equation $(K_{i(x+iy)}(a))^2$ should read $(K_{ix}(a))^2$).

Remark 1. Formula (2) can also be derived by first applying the Pfaff-Kummer transformation formula [3, (1.5.5)] to the ${}_2F_1$ series in (1) with the additional restriction that $|(1-t)/t| < 1$, using the binomial theorem to expand each of the t^{-j-1} powers of t in powers of $1-t$, changing the order of summation, and then employing analytic continuation to remove the $|(1-t)/t| < 1$ restriction.

Remark 2. From (6) and an extension of the derivation of (2) it follows that the generating function (cf. [4, (1.3.6)], [5, p. 398])

$$(1-t)^{x-c}\,{}_2F_1(a+x, b+x; a+b; t)$$

$$= \sum_{n=0}^{\infty} \frac{S_n(-x^2; a, b, c)}{n!\,(a+b)_n}\, t^n, \quad 0 \le t < 1,$$

is an absolutely monotonic function of x when $c \ge 0$ and either $a, b > 0$ or a, b are complex conjugates with positive real parts, and that the generating function (cf. [2, 19.10 (25)], [4, (1.3.9)])

$$e^t\,{}_2F_2(a+x, a-x; a+b, a+c; -t)$$

$$= \sum_{n=0}^{\infty} \frac{S_n(-x^2; a, b, c)}{n!\,(a+b)_n\,(a+c)_n}\, t^n, \quad 0 \le t < \infty,$$

is an absolutely monotonic function of x when $a, b, c > 0$.

Remark 3. Similarly, additional absolutely monotonic functions can be obtained by using the generating functions for the Wilson $W_n(x^2; a, b, c, d)$ and the Askey-Wilson $p_n(x; a, b, c, d|q)$ polynomials in [4, §1.1 and §3.1], and their limit cases, along with suitable changes in variables. In particular, it follows from [4, (1.1.2) and (1.1.6)], [6, (1.2) and (2.4)] that the generating function

$${}_2F_1(a+x, b+x; a+b; t)\,{}_2F_1(c-x, d-x; c+d; t)$$

$$= \sum_{n=0}^{\infty} \frac{W_n(-x^2; a, b, c, d)}{n!\,(a+b)_n\,(c+d)_n}\, t^n, \quad 0 \le t < 1,$$

is an absolutely monotonic function of x when $\mathrm{Re}(a, b, c, d) > 0$ and non-real parameters occur in conjugate pairs with $a+b > 0$ and $c+d > 0$.

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ftp://unvie6.un.or.at/siam/opsf_new/koekoek_swarttouw/koekoek_swarttouw1.ps

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