Fractional integration for Laguerre expansions

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(To appear in Methods and Applications of Analysis)

Key words. Fractional integration, multiplier conditions, Laguerre polynomials

AMS(MOS) subject classifications. 33C45, 42A45, 42C10

1 Introduction

The aim of this note is to provide a fractional integration theorem in the framework of Laguerre expansions. The method of proof consists of establishing an asymptotic estimate for the involved kernel and then applying a method of Hedberg [5]. We combine this result with sufficient (p,p) multiplier criteria of Stempak and Trebels [10]. The resulting sufficient (p,q) multiplier criteria are comparable with necessary ones of Gasper and Trebels [3].

Our notation is essentially that in [10]. Thus we consider the Lebesgue spaces

$$L^p_{v(\gamma)} = \{f : \| f \|_{L^p_{v(\gamma)}} = (\int_0^\infty |f(x)|^p x^\gamma \, dx)^{1/p} < \infty\} \,, \quad 1 \le p < \infty, \quad \gamma > -1,$$

and define the Laguerre function system $\{l_k^\alpha\}$ by

$$l_k^\alpha(x) = (k!/\Gamma(k+\alpha+1))^{1/2} e^{-x/2} L_k^\alpha(x), \quad \alpha > -1, \quad k \in \mathbf{N}_0.$$

This system is an orthonormal basis in $L^2(\mathbf{R}_+,\, x^\alpha dx)$ and for $\gamma < p(\alpha+1)-1$ we can associate to any $f \in L^p_{v(\gamma)}$ the Laguerre series

$$f(x) \sim \sum_{k=0}^\infty a_k l_k^\alpha(x), \qquad a_k = \int_0^\infty f(x) l_k^\alpha(x) x^\alpha dx.$$

It is convenient to introduce the vector space

$$E = \{f(x) = p(x)e^{-x/2} \;:\; 0 \le x < \infty, \;\; p(x) \text{ a polynomial}\}$$

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which is dense in $L^p_{v(\gamma)}$. We note that $f \in E$ has only finitely many non-zero Fourier-Laguerre coefficients. Analogous to the definition of the Hardy and Littlewood fractional integral operator for Fourier series (see [12, Chap. XII, Sec. 8], we define a fractional integral operator I_σ, $\sigma > 0$, for Laguerre expansions by

$$I_\sigma f(x) = \sum_{k=0}^{\infty} (k+1)^{-\sigma} a_k l_k^\alpha(x)\,, \qquad f \in E.$$

Observing that the l_k^α are eigenfunctions with eigenvalues λ_k of the differential operator

$$L = -\Big(x\frac{d^2}{dx^2} + (\alpha+1)\frac{d}{dx} - \frac{x}{4}\Big)\,, \quad \lambda_k = k + (\alpha+1)/2,$$

(see [11, (5.1.2)]) one realizes that I_1 is an integral operator essentially inverse to L (see the following Remark 2). As we will see in Section 2 the fractional integral $I_\sigma f$ can be interpreted as a twisted convolution of f with a function

$$g_\sigma(x) \sim \Gamma(\alpha+1) \sum_{k=0}^{\infty} (k+1)^{-\sigma} L_k^\alpha(x) e^{-x/2}\,. \tag{1}$$

From Theorem 3.1 in [3, II] it easily follows that $g_\sigma \in L^1_{v(\gamma)}$ when $\alpha - \gamma < \sigma$, so that, by the convolution theorem of Görlich and Markett [4], I_σ extends to a bounded operator from $L^p_{v(\gamma)}$ to $L^p_{v(\gamma)}$ when $0 \le \alpha p/2 \le \gamma \le \alpha$, $1 \le p \le \infty$. Our main result is

Theorem 1.1 *Let $\alpha \ge 0$, $1 < p \le q < \infty$. Assume further that $0 < \sigma < \alpha + 1$, $a < (\alpha+1)/p'$, $b < (\alpha+1)/q$, $a + b \ge 0$. Then*

$$||I_\sigma f||_{L^q_{v(\alpha - bq)}} \le C||f||_{L^p_{v(\alpha+ap)}}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\sigma - a - b}{\alpha+1}.$$

Remarks. 1) Observe that the upper bounds for a and b imply lower bounds for b and a, respectively.

2) By the above it is clear that the sequence $\{(k+1)^{-\sigma}\}$ which generates the fractional integral can be replaced by any sequence $\{\Omega_\sigma(k)\}$ satisfying

$$\Omega_\sigma(k) = \sum_{j=0}^{J} c_j\,(k+1)^{-\sigma-j} + O((k+1)^{-\sigma-J})$$

for sufficiently large J, say $J \ge \alpha + 2$, thus in particular obtaining the same result for the sequence $\{\Gamma(k+1)/\Gamma(\sigma+k+1)\}$ – see [7].

3) A weaker version of Theorem 1.1 (e.g. in the case $a = b = 0$) can easily be deduced by the following argument. By a slight modification of the proof of Theorem 3.1 in [3, II] we have:

Let $\alpha > -1$ *and* $N \in \mathbf{N}_0$, $N > (2\alpha+2)(1/r-1/2)-1/3$. *If* $\{f_k\}$ *is a bounded sequence with* $\lim_{k\to\infty} f_k = 0$ *and*

$$\sum_{k=0}^{\infty}(k+1)^{N+(\alpha+1)/r'}|\Delta^{N+1}f_k| \le K_r(f) < \infty, \quad 1 \le r < \infty,$$

then there exists a function $f \in L^r_{v(\alpha)}$ *with*

$$\|f\|_{L^r_{v(\alpha)}} \le CK_r(f), \quad f(x) \sim \sum_{k=0}^{\infty} f_k L_k^\alpha(x) e^{-x/2}.$$

This applied to the sequence $\{(k+1)^{-\sigma}\}$ gives, by Young's inequality (see [4]),

$$||I_\sigma f||_{L^q_{v(\alpha)}} \le C||f||_{L^p_{v(\alpha)}}, \quad \frac{1}{q} > \frac{1}{p} - \frac{\sigma}{\alpha+1},$$

where $\alpha \ge 0$, $\sigma > 0$ and $1 \le p,\ q \le \infty$.

Next we indicate how Theorem 1.1 can be used to gain some insight into the structure of $M^{p,q}$–Laguerre multipliers. For the sake of simplicity let us restrict ourselves to the case $\gamma = \alpha$. Consider a sequence $m = \{m_k\}$ of numbers and associate to m the operator

$$T_m f(x) = \sum_{k=0}^{\infty} m_k a_k l_k^\alpha(x), \quad f \in E.$$

The sequence m is called a bounded (p,q)-multiplier, notation $m \in M^{p,q}_{\alpha,\alpha}$, if

$$||\, m\, ||_{M^{p,q}_{\alpha,\alpha}} := \inf\{C :\ ||T_m f||_{L^q_{v(\alpha)}} \le C\|f\|_{L^p_{v(\alpha)}} \quad \text{for all } f \in E\}$$

is finite. If $p \le 2 \le q$, then sufficient conditions follow at once in the following way: observe that $M^{2,2}_{\alpha,\alpha} = l^\infty$, choose $\sigma_0, \sigma_1 \ge 0$ such that $I_{\sigma_0} : L^p \to L^2$, $I_{\sigma_1} : L^2 \to L^q$, $\sigma_0 + \sigma_1 = \sigma$, and hence

$$||\, m\, ||_{M^{p,q}_{\alpha,\alpha}} \le \|k^{-\sigma_0}\|_{M^{p,2}_{\alpha,\alpha}} ||\{k^\sigma m_k\}||_{M^{2,2}_{\alpha,\alpha}} \|k^{-\sigma_1}\|_{M^{2,q}_{\alpha,\alpha}}\ ;$$

in particular, $m \in M^{p,q}_{\alpha,\alpha}$ when $\{k^\sigma m_k\} \in l^\infty$. Thus when $1 < p \le 2 \le q < \infty$, and σ is as in Theorem 1.1, $\{(-1)^k(k+1)^{-\sigma}\} \in M^{p,q}_{\alpha,\alpha}$ generates a bounded operator which does not fall under the scope of the above introduced fractional integral operators. To formulate a Corollary based on a combination of Theorem 1.1 and the multiplier result in [10] we need the notion of a difference operator Δ^s of fractional order s given by

$$\Delta^s m_k = \sum_{j=0}^{\infty} A_j^{-s-1} m_{k+j}\,, \quad A_j^t = \frac{\Gamma(j+t+1)}{j!\,\Gamma(t+1)}, \quad t \in \mathbf{R},$$

whenever the sum converges. In view of the remark concerning the case $p < 2 < q$ and on account of duality we may restrict ourselves to the case $1 < p < q < 2$.

Corollary 1.2 *If $\alpha \geq 0$, $1 < p < q < 2$, and $s > \max\{(2\alpha+2)(1/q-1/2),\, 1\}$ then, for some constant C independent of the sequence $\{m_k\}$, there holds*

$$\|\{m_k\}\|^2_{M^{p,q}_{\alpha,\alpha}} \leq C\left(\|\{k^\sigma m_k\}\|^2_\infty + \sup_N \sum_{k=N}^{2N} |(k+1)^{s+\sigma}\Delta^s m_k|^2 (k+1)^{-1}\right).$$

The proof follows as in [2]. The result itself should be compared with the corresponding necessary condition in [3, I] which also shows that $\{(-1)^k(k+1)^{-\sigma}\} \notin M^{p,q}_{\alpha,\alpha}$ provided $1 < p < q < 2$ and α is sufficiently large.

The plan of the paper is as follows. In Section 2 we derive an asymptotic estimate of the function g_σ defined above by (1). Then the twisted generalized convolution is used to dominate $I_\sigma f$ by a generalized Euclidean convolution of g_σ with f. The latter's mapping behavior is discussed by a method of Hedberg [5] which uses maximal functions; thus giving Theorem 1.1 in the standard weight case $a = b = 0$. In Section 3 we extend this result to some power weights, modifying an argument in Stein and Weiss [8].

2 Proof of the standard weight case.

We start by deriving the required asymptotic estimate and showing

Lemma 2.1 *Let $\alpha > -1$. Then, for $x > 0$ and $0 < \sigma < \alpha + 1$, there holds*

$$|g_\sigma(x)| \leq C x^{\sigma-\alpha-1}. \tag{2}$$

Proof. First note that by subordination (see [3, I], p. 1234) there holds for $N > \alpha + 2$, $N \in \mathbf{N}$,

$$g_\sigma(x) = C\sum_{k=0}^{\infty}\left(\Delta^N (k+1)^{-\sigma}\right) L_k^{\alpha+N}(x)e^{-x/2}.$$

Then the assertion (2) follows after it is proved that

$$\sum_{k=0}^{\infty}(k+1)^{-\sigma-N}\,|x^{\alpha+1-\sigma}L_k^{\alpha+N}(x)e^{-x/2}| \leq C \tag{3}$$

uniformly in $x > 0$. With the notation

$$\mathcal{L}_k^\alpha(x) = (k!/\Gamma(k+\alpha+1))^{1/2} x^{\alpha/2} e^{-x/2} L_k^\alpha(x), \quad k \in \mathbf{N}_0.$$

this is equivalent to

$$\sum_{k=0}^{\infty}(k+1)^{(\alpha-N)/2-\sigma}\,|\mathcal{L}_k^{\alpha+N}(x)| \leq C x^{\sigma+(N-\alpha-2)/2} \tag{4}$$

uniformly in $0 < x < \infty$. We will make use of the pointwise estimates for the Laguerre functions in [1, Sec. 2]:

$$|\mathcal{L}_k^{\alpha+N}(x)| \leq C \begin{cases} (x(k+1))^{(\alpha+N)/2} & if \quad 0 \leq x \leq c/(k+1), \\ (x(k+1))^{-1/4} & if \quad c/(k+1) \leq x \leq d(k+1) \end{cases} \tag{5}$$

for fixed positive constants c and d. These and two further estimates in [6, (2.5)] imply that

$$\sup_{x>0} |\mathcal{L}_k^{\alpha+N}(x)| \leq C, \quad k \in \mathbf{N}_0, \tag{6}$$

and so it is obvious that (6) implies (4) for $x \geq 1$.

Therefore, decomposing dyadically the interval $(0,1)$ it suffices to check that

$$\sum_{k=0}^{\infty}(k+1)^{(\alpha-N)/2-\sigma} |\mathcal{L}_k^{\alpha+N}(x)| \leq C(2^j)^{\sigma+(N-\alpha-2)/2}$$

provided $2^j \leq x \leq 2^{j+1},\ j < 0$. Using the first line of (5) we get

$$\sum_{k=0}^{2^{-j}}(k+1)^{(\alpha-N)/2-\sigma} |\mathcal{L}_k^{\alpha+N}(x)| \leq C(2^j)^{(\alpha+N)/2}\sum_{k=0}^{2^{-j}}(k+1)^{\alpha-\sigma} \leq C(2^j)^{\sigma+(N-\alpha-2)/2},$$

while the second line of (5) gives

$$\sum_{k=2^{-j}}^{\infty} k^{(\alpha-N)/2-\sigma} |\mathcal{L}_k^{\alpha+N}(x)| \leq C2^{-j/4} \sum_{k=2^{-j}}^{\infty} k^{(\alpha-N)/2-\sigma-1/4} \leq C(2^j)^{\sigma+(N-\alpha-2)/2}.$$

This completes the proof of Lemma 2.1.

As already mentioned we will apply Hedberg's method which involves maximal functions. To make use of the corresponding results in [9] we switch to the system

$$\psi_k^\alpha(x) = (2k!/\Gamma(k+\alpha+1))^{1/2} e^{-x^2/2} L_k^\alpha(x^2), \quad k \in \mathbf{N}_0,$$

which is obviously orthonormal on $L^2(\mathbf{R}_+, d\mu_\alpha),\ d\mu_\alpha(x) = x^{2\alpha+1}dx,\ \alpha > -1$. For the sake of simplicity we write the norm on $L^p(\mathbf{R}_+, d\mu_\alpha)$ as

$$\|F\|_p = \left(\int_0^\infty |F(x)|^p d\mu_\alpha(x)\right)^{1/p}.$$

We adopt the notion of the twisted generalized convolution on $L^1(\mathbf{R}_+, d\mu_\alpha)$ from [9]

$$F \times G(x) = \int_0^\infty \tau_x F(y) G(y)\, d\mu_\alpha(y),$$

where the twisted generalized translation operator τ_x is given by

$$\tau_x F(y) = \frac{\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+1/2)} \int_0^\pi F((x,y)_\theta)\mathcal{J}_{\alpha-1/2}(xy\sin\theta)(\sin\theta)^{2\alpha}\,d\theta,$$

$\mathcal{J}_\beta(x) = \Gamma(\beta+1)J_\beta(x)/(x/2)^\beta$, J_β denoting the Bessel function of order β, and

$$(x,y)_\theta = (x^2+y^2-2xy\cos\theta)^{1/2}.$$

With respect to the system $\{\psi_k^\alpha\}$ this convolution has the following transform property: if $F \sim \sum c_k\psi_k^\alpha$ and $F\times G \sim \sum c_k d_k \psi_k^\alpha$, then $G(x) \sim \Gamma(\alpha+1)\sum d_k L_k^\alpha(x^2)e^{-x^2/2}$. If we set $f(y^2) = F(y)$, $g_\sigma(y^2) = G_\sigma(y)$, we see that

$$|I_\sigma f(x^2)| = |F\times G_\sigma(x)|.$$

Apart from Lemma 2.1 the proof of Theorem 1.1 will be based on the fact that for $\alpha \geq 0$ and suitable F and G

$$|F\times G| \leq |F| * |G|\,,$$

which follows at once from the definition of the generalized Euclidean $*$-convolution

$$F * G(x) = \int_0^\infty \tau_x^E F(y)G(y)\,d\mu_\alpha(y)$$

with associated generalized Euclidean translation

$$\tau_x^E F(y) = \int_0^\pi F((x,y)_\theta)\,d\nu_\alpha(\theta), \quad d\nu_\alpha(\theta) = \frac{\Gamma(\alpha+1)}{\pi^{1/2}\Gamma(\alpha+1/2)}(\sin\theta)^{2\alpha}d\theta\,.$$

Therefore we restrict ourselves to fractional integrals defined via the generalized Euclidean convolution.

Theorem 2.2 *Let* $1<p<q<\infty$, $\alpha > -1/2$, $K_\sigma(x) = x^{2(\sigma-\alpha-1)}$. *Then*

$$\|F * K_\sigma\|_q \leq C\|F\|_p\,, \quad \frac{1}{q} = \frac{1}{p} - \frac{\sigma}{\alpha+1}\,.$$

By the above it is clear that Theorem 1.1 for the case $a=b=0$ follows from Theorem 2.2.

Proof. Following Hedberg [5] we want to estimate $F * K_\sigma(x)$ pointwise by a suitable maximal function which in this setting turns out to be (see Stempak [9, p. 138])

$$F^*(x) = \sup_{\varepsilon>0} \varepsilon^{-(2\alpha+2)} \int_0^\varepsilon \tau_x^E(|F|)(y)d\mu_\alpha(y)$$

with the usual boundedness property $||F^*||_r \leq C||F||_r, \quad 1 < r \leq \infty$. Now there holds

$$|F * K_\sigma(x)| \leq C \left(\int_0^\delta + \int_\delta^\infty \right) \tau_x^E(|F|)(y) y^{2(\sigma-\alpha-1)} d\mu_\alpha(y) = J_1 + J_2,$$

where $\delta > 0$ will be chosen later appropriately. Clearly,

$$J_1 = \sum_{k=0}^{\infty} \int_{2^{-k-1}\delta}^{2^{-k}\delta} \cdots \leq C \sum_{k=0}^{\infty} (2^{-k}\delta)^{2\sigma} (2^{-k}\delta)^{-2\alpha-2} \int_{2^{-k-1}\delta}^{2^{-k}\delta} \tau_x^E(|F|)(s) s^{2\alpha+1} ds$$

$$\leq C\delta^{2\sigma} \sum_{k=0}^{\infty} 2^{-k2\sigma} (2^{-k}\delta)^{-2\alpha-2} \int_0^{\delta 2^{-k}} \tau_x^E(|F|)(s) s^{2\alpha+1} ds \leq C\delta^{2\sigma} F^*(x).$$

On the other hand, by Hölder's inequality,

$$J_2 = \int_\delta^\infty \tau_x^E(|F|)(s) s^{2(\sigma-\alpha-1)} s^{2\alpha+1} ds \leq ||\tau_x^E(|F|)||_p \left(\int_\delta^\infty s^{2(\sigma-\alpha-1)p'} s^{2\alpha+1} ds \right)^{1/p'}$$

$$\leq C\delta^{2\sigma-(2\alpha+2)/p} ||F||_p \ ,$$

since τ_x^E are contractions on $L^p(\mathbf{R}_+, d\mu_\alpha)$. Hence

$$|F * K_\sigma(x)| \leq C \left(\delta^{2\sigma} F^*(x) + \delta^{2\sigma-(2\alpha+2)/p} ||F||_p \right).$$

Choosing $\delta = (F^*(x)/||F||_p)^{-p/(2\alpha+2)}$ (where $||F||_p \neq 0$ is assumed) we obtain

$$|F * K_\sigma(x)| \leq C F^*(x)^{1-\sigma p/(\alpha+1)} \|F\|_p^{\sigma p/(\alpha+1)}$$

and then $\|F * K_\sigma\|_q \leq C||F||_p$ due to the inequality for the maximal function with $r = q(1 - \sigma p/(\alpha+1)) = p > 1$. This finishes the proof of Theorem 2.2.

3 Extension to power weights

The proof of Theorem 1.1 in the general case follows along the lines in Section 2 from

Theorem 3.1 *Let $\alpha > -1/2$, $0 < \sigma < \alpha+1$ and $a < (\alpha+1)/p'$, $b < (\alpha+1)/q$, with $a+b \geq 0$. If $1 < p \leq q < \infty$, then*

$$\|K_\sigma * F(x)\, x^{-2b}\|_q \leq C\|F(x)\, x^{2a}\|_p, \quad \frac{1}{q} = \frac{1}{p} - \frac{\sigma - a - b}{\alpha + 1}.$$

Proof. An equivalent version of the inequality above is

$$\left(\int_0^\infty |Sf(x)|^q d\mu_\alpha(x)\right)^{1/q} \le C \left(\int_0^\infty |f(x)|^p d\mu_\alpha(x)\right)^{1/p}$$

where $f(x) = x^{2a}F(x)$, $Sf(x) = \int_0^\infty K(x,y)f(y)d\mu_\alpha(y)$ and

$$K(x,y) = x^{-2b}\left(\int_0^\pi (x,y)_\theta^{2(\sigma-\alpha-1)} d\nu_\alpha(\theta)\right) y^{-2a}.$$

We first consider the case $p = q$. Then $\sigma = a + b$ and, therefore, the kernel $K(x,y)$ is homogeneous of degree $-(2\alpha+2)$: $K(rx, ry) = r^{-(2\alpha+2)}K(x,y)$, $r > 0$. It now suffices to check that (cf. Section 2 of [10])

$$\int_0^\infty K(1,y)y^{-(2\alpha+2)/p} d\mu_\alpha(y) < \infty.$$

We first note that the function $K(1,y)$ has at most an integrable singularity at $y = 1$, since for $1/2 \le y \le 2$ we have

$$\int_0^\pi \frac{(\sin\theta)^{2\alpha} d\theta}{((1-y)^2 + 4y\sin^2(\theta/2))^{\alpha+1-\sigma}} \le C(1 + |1-y|^{\sigma-1}).$$

To deal with the singularity at 0 we note that for $y < 1/2$ we have $(1,y)_\theta \approx 1$ and therefore $\int_0^\pi (1,y)_\theta^{2(\sigma-\alpha-1)} d\nu_\alpha(\theta) \approx 1$ as well. Hence $K(1,y) \le Cy^{-2a}$ and thus

$$\int_0^{1/2} K(1,y)y^{-(2\alpha+2)/p} d\mu_\alpha(y) < \infty, \quad a < (\alpha+1)/p'.$$

For $y > 2$ we have $(1,y)_\theta \approx y$, $\int_0^\pi (1,y)_\theta^{2(\sigma-\alpha-1)} d\nu_\alpha(\theta) \approx y^{2(\sigma-\alpha-1)}$ and hence

$$\int_2^\infty K(1,y)y^{-(2\alpha+2)/p} d\mu_\alpha(y) < \infty, \quad b < (\alpha+1)/q,$$

since $\sigma < (\alpha+1)/p$.

Now, consider the case $1 < p < q < \infty$. We use still another equivalent version of the inequality to be proved, namely

$$\int_0^\infty \int_0^\infty K(x,y)f(y)g(x)\, d\mu_\alpha(y)\, d\mu_\alpha(x) \le C||f||_p||g||_{q'}\ ,$$

assuming for simplicity that f and g are nonnegative. Writing $\mathbf{R}_+ \times \mathbf{R}_+ = D_1 \cup D_2 \cup D_3$ where

$$D_1 = \{y/2 \le x \le 2y\},\ D_2 = \{2y < x\},\ D_3 = \{x < y/2\}$$

it suffices to check for $i=1,2,3$ that

$$I_i=\iint_{D_i} K(x,y)f(y)g(x)\,d\mu_\alpha(y)\,d\mu_\alpha(x)\le C\|f\|_p\|g\|_{q'}\ .$$

Consider I_1 first. Since $a+b\ge 0$, for $x,y\in D_1$ and $\theta\in(0,\pi)$ we have

$$((x,y)_\theta)^{2(a+b)}\le Cx^{2(a+b)}\le Cx^{2b}y^{2a};$$

so

$$I_1\le\iint_{D_1}\widetilde{K}(x,y)f(y)g(x)\,d\mu_\alpha(y)\,d\mu_\alpha(x)$$

with $\widetilde{K}(x,y)=\tau_x^E K_{\sigma-a-b}(y)$. Hence we are reduced to showing that

$$\int_0^\infty K_{\sigma-a-b}*f(x)\,g(x)\,d\mu_\alpha(x)\le||f||_p||g||_{q'},$$

which is implied by Theorem 2.2.

To estimate I_2 and I_3 we need the following lemma.

Lemma 3.2 *Let $\alpha>-1$ and $V_\delta f(x)=x^{2(\delta-\alpha-1)}\int_0^x f(y)y^{-2\delta}d\mu_\alpha(y)$, $\delta<(\alpha+1)/p'$. Then*

$$||V_\delta f||_p\le C||f||_p\,,\quad |V_\delta f(x)|\le C\,x^{-(2\alpha+2)/p}||f||_p\,.$$

Proof. We have that $V_\delta f(x)=\int_0^\infty L(x,y)f(y)d\mu_\alpha(y)$, where $L(x,y)=x^{2(\delta-\alpha-1)}y^{-2\delta}$ for $y<x$ and equals 0 otherwise. Clearly L is homogeneous of degree $-(2\alpha+2)$. Therefore the norm inequality is implied by

$$\int_0^\infty L(1,y)y^{-(2\alpha+2)/p}d\mu_\alpha(y)=\int_0^1 y^{-2\delta-(2\alpha+2)/p+2\alpha+1}\,dy<\infty$$

which is finite since $\delta<(\alpha+1)/p'$. For the pointwise estimate we simply write

$$|V_\delta f(x)|\le x^{2(\delta-\alpha-1)}\int_0^x|f(y)|\,y^{-2\delta}d\mu_\alpha(y)$$

$$\le Cx^{2(\delta-\alpha-1)}\left(\int_0^x y^{-2\delta p'}d\mu_\alpha(y)\right)^{1/p'}||f||_p\le Cx^{-(2\alpha+2)/p}||f||_p$$

which finishes the proof of the lemma.

Estimating I_2, we note that for $x,y\in D_2$ and all θ, $0<\theta<\pi$, there holds $x<2(x-y)\le 2((x-y)^2+2xy(1-\cos\theta))^{1/2}=2(x,y)_\theta$; hence

$$I_2\le C\int_0^\infty x^{2(\sigma-\alpha-1-b)}g(x)\left(\int_0^x f(y)\,y^{-2a}d\mu_\alpha(y)\right)d\mu_\alpha(x)$$

$$= C\int_0^\infty g(x)x^{2(\sigma-a-b)}V_af(x)\,d\mu_\alpha(x) \le C||g||_{q'}||x^{2(\sigma-a-b)}V_af||_q\,.$$

It now suffices to estimate the latter L^q norm by $||f||_p$. We have

$$\int_0^\infty |x^{2(\sigma-a-b)}V_af(x)|^q d\mu_\alpha(x) = \int_0^\infty |V_af(x)|^p|V_af(x)|^{q-p}x^{2(\sigma-a-b)q}d\mu_\alpha(x)\,.$$

Since $a < (\alpha+1)/p'$, by the pointwise estimate in the above lemma

$$|V_af(x)| \le Cx^{-(2\alpha+2)/p}||f||_p\,;$$

hence

$$|V_af(x)|^{q-p}x^{2(\sigma-a-b)q} \le C||f||_p^{q-p}$$

due to the identity from assumptions. Hence

$$\int_0^\infty |x^{2(\sigma-a-b)}V_af(x)|^q d\mu_\alpha(x) \le C||f||_p^{q-p}\int_0^\infty |V_af(x)|^p d\mu_\alpha(x) \le C||f||_p^q\,.$$

by the norm inequality from Lemma 3.2. The estimate of I_3 is similar and we omit it. The proof of the theorem is complete.

After submitting this paper the authors became acquainted with the following result of Kanjin and Sato [The Hardy-Littlewood theorem on fractional integration for Laguerre series, Proc. Amer. Math. Soc., to appear]:

$$\|I_\sigma f\|_{L^q_{v(\alpha q/2)}} \le C\|f\|_{L^p_{v(\alpha p/2)}}, \quad 0 < \frac{1}{q} = \frac{1}{p} - \sigma, \quad p > 1, \quad \alpha \ge 0,$$

whereas for $-1 < \alpha < 0$ there occurs $(1+\frac{\alpha}{2})^{-1} < p$, $q < -\frac{2}{\alpha}$ as additional restriction. Combining the above Theorem 1.1 in the case $a = b = \alpha = 0$ with Kanjin's transplantation theorem (cf. [10]) one at once recovers the above stated result of Kanjin and Sato .

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