# A lower estimate for the Lebesgue constants of linear means of Laguerre expansions

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*Dedicated to P. L. Butzer on the occasion of his 70th birthday in gratitude*

(Oct. 14, 1997 version)

**Abstract.** S. G. Kal'neĭ derived in [5], [6] a quite sharp necessary condition for the multiplier norm of a finite sequence in the setting of Fourier-Jacobi series on $L^1$ with "natural weight" (which ensures a nice convolution structure). In this paper, Kalneĭ's problem is considered in the setting of Laguerre series on weighted $L^1$-spaces; the admitted scale of weights contains in particular the appropriate "natural weights" occurring in transplantation and convolution.

**Key words.** Laguerre polynomials, linear means, Lebesgue constants, weighted Lebesgue spaces

**AMS(MOS) subject classifications.** Primary: 33C45, 42A45, 42C10; Secondary: 39A70

## 1 Introduction

The purpose of this paper is to derive a good lower bound for the $L^1$-Laguerre multiplier norm of a finite sequence $m = \{m_k\}$, $m_k = 0$ when $k \geq n+1$. Such sequences occur when one considers linear summability methods generated by a lower triangular numerical matrix $\Lambda = \{\lambda_k^n\}$. A very important example is the Cesàro method for which the general necessary criteria in [9], [2] only give a constant as a lower bound at the critical index whereas discussing the Cesàro means (at the critical index) directly gives a logarithmic divergence (see [3], [4]). To become more precise let us first introduce some notation. Consider the Lebesgue spaces

$$L^p_{w(\gamma)} = \{f:\ \|f\|_{p,\gamma} = (\int_0^\infty |f(x)e^{-x/2}|^p x^\gamma\, dx)^{1/p} < \infty\}\ , \quad 1 \leq p < \infty,$$

$$L^\infty_{w(\gamma)} = \{f:\ \|f\|_{\infty,\gamma} = \operatorname{ess\,sup}_{x>0} |f(x)e^{-x/2}x^{\alpha-\gamma}| < \infty\}, \quad p = \infty,$$

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where $\alpha \geq \gamma > -1$. Let $L_n^\alpha(x)$, $n \in \mathbf{N}_0$, denote the classical Laguerre polynomials (see Szegö [10, p. 100]) and set

$$R_n^\alpha(x) = L_n^\alpha(x)/L_n^\alpha(0), \qquad L_n^\alpha(0) = A_n^\alpha = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

Then one can associate to $f$ its formal Laguerre series

$$f(x) \sim (\Gamma(\alpha+1))^{-1} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\alpha(x),$$

where the Fourier-Laguerre coefficients of $f$ are defined by

$$\hat{f}_\alpha(n) = \int_0^\infty f(x) R_n^\alpha(x) x^\alpha e^{-x}\, dx \tag{1}$$

(if the integrals exist). A sequence $m = \{m_k\}_{k \in \mathbf{N}_0}$ is called a (bounded) multiplier on $L^p_{w(\gamma)}$, notation $m \in M^p_{\alpha;\gamma}$, if

$$\| \sum_{k=0}^{\infty} m_k \hat{f}_\alpha(k) L_k^\alpha \|_{p,\gamma} \leq C \| \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\alpha \|_{p,\gamma} \tag{2}$$

for all polynomials $f$; the smallest constant $C$ for which this holds is called the multiplier norm $\|m\|_{M^p_{\alpha,\gamma}}$.

We are interested in good lower estimates of $\|m\|_{M^1_{\alpha,\gamma}}$ for **finite** sequences $m = \{m_k\}$, $m_k = 0$ when $k \geq n+1$. By the definition of the multiplier norm we have

$$\|m\|_{M^1_{\alpha,\gamma}} = \sup_{\|f\|_{1,\gamma} \leq 1} \| \sum_{k=0}^{n} m_k \hat{f}_\alpha(k) L_k^\alpha \|_{1,\gamma} \geq C(n+1)^{\gamma-\alpha} \| \sum_{k=0}^{n} m_k L_k^\alpha \|_{1,\gamma}. \tag{3}$$

Here the particular test function $\Phi_n$ is given via its coefficients $(\Phi_n)\hat{}_\alpha(k) = \phi(k/n)$ where $\phi$ is a smooth cut-off function with $\phi(t) = 1$ for $0 \leq t \leq 1$ and $= 0$ for $t \geq 2$. For this function there holds by [2], formula (9) in I, $\|\Phi_n\|_{1,\gamma} \leq C(n+1)^{\alpha-\gamma}$, $\alpha \geq \gamma > -1$, hence (3).
Generic positive constants that are independent of the parameter $n$ and of the sequence $m$ will be denoted by $C$. Our main result now reads

THEOREM 1. *Suppose $\alpha \geq 0$, $\alpha/2 \leq \gamma \leq \alpha$. Then for any finite sequence $m = \{m_k\}$, $m_k = 0$ for $k \geq n+1$, there holds*

$$\|m\|_{M^1_{\alpha,\gamma}} \geq C(n+1)^{\gamma-\alpha} \sum_{k=0}^{n} |m_k| \frac{(k+1)^{\gamma+1/2}}{(n+1-k)^{2\gamma-\alpha+3/2}}, \tag{4}$$

*where $C$ is a constant independent of $n$.*

In the following remarks we discuss this result for the standard weight $\gamma = \alpha/2$, which is the natural setting for transplantation theorems (see Kanjin [7]), and the weight $\gamma = \alpha$, which is the natural setting for a nice convolution structure (see Görlich and Markett [4]).

REMARKS. 1) Taking only the $(k = n)$–term in (4) and $\gamma = \alpha$ one arrives at the Cohen type inequality (1.8) in [9], whereas $\gamma = \alpha/2$ leads to (1.9) in [9].

2) The Cesàro means of order $\delta$ are generated by the matrix $\Lambda = (\lambda_k^n)$ where $\lambda_k^n = A_{n-k}^{\delta}/A_n^{\delta}$. If one evaluates the $\sum_{[n/2]}^{n} \ldots$–portion in (4) in the case of the critical index $\delta_c = 1/2$ when $\gamma = \alpha/2$ or $\delta_c = \alpha + 1/2$ when $\gamma = \alpha$ one obtains

$$\|\{A_{n-k}^{\delta_c}/A_n^{\delta_c}\}\|_{M^1_{\alpha,\gamma}} \geq C \log(n+1)\,, \qquad \alpha \geq 0\,.$$

For $\gamma = \alpha/2$ this is directly computed in [3], for $\gamma = \alpha$ in [4].
In the case $0 \leq \delta < \delta_c = 2\gamma - \alpha + 1/2$ already the $(k = n)$–term of the sum (cf. the above mentioned Cohen type inequalities) leads to the estimate (for $\gamma = \alpha/2$ see [3], for $\gamma = \alpha$ [4])

$$\|\{A_{n-k}^{\delta}/A_n^{\delta}\}\|_{M^1_{\alpha,\gamma}} \geq C(n+1)^{\delta_c - \delta}\,, \quad 0 \leq \delta < \delta_c\,.$$

3) Obviously, by omitting the terms on the right side of (4) with $k < [n/2]$, there holds

$$\|m\|_{M^1_{\alpha,\gamma}} \geq C(n+1)^{\delta_c} \sum_{k=[n/2]}^{n} |m_k|(n+1-k)^{-\delta_c - 1}\,. \tag{5}$$

which is equivalent to (4) since $M^1_{\alpha,\gamma} \subset l^\infty$ (just choose $f = L_k^\alpha$ in (2)). Condition (5) may be compared with the necessary conditions given in [2].

For the **proof of the Theorem** we follow the lines of Kal'neĭ and first observe that by the converse of Hölder's inequality we may continue the estimate (3) as follows

$$\|m\|_{M^1_{\alpha,\gamma}} \geq C(n+1)^{\gamma - \alpha} \sup_{\|g\|_{\infty,\gamma} \leq 1} \int_0^\infty \sum_{k=0}^{n} m_k L_k^\alpha(x) g(x) e^{-x} x^\alpha dx\,. \tag{6}$$

If we choose a particular $g$ we make the right hand side of (6) smaller. Consider the test function $g = g_n$,

$$g_n(x) = \sum_{j=0}^{n} (\operatorname{sgn} m_j)\, \Delta^N_{2(n+1-j)} R_j^\alpha(x) \frac{(j+1)^{\gamma+1/2}}{(n+1-j)^{2\gamma-\alpha+3/2}}\,, \tag{7}$$

where $\Delta_k R_j^\alpha = R_j^\alpha - R_{j+k}^\alpha$, $\Delta^N = \Delta(\Delta^{N-1})$, and $N \in \mathbf{N}$ is so large that $N - 1 \leq 2\gamma - \alpha + 1/2 < N$. Suppose that the lemma below holds, then $g_n \in L^\infty_{w(\gamma)}$ is obviously true and the assertion of the Theorem immediately follows by the orthogonality of the Laguerre polynomials.
Thus there only remains to prove the following result.

LEMMA. *Suppose that* $0 \le \alpha/2 \le \gamma \le \alpha$ *and set*

$$f_n^{\alpha,\gamma}(x) = \sum_{j=0}^{n} |\Delta^N_{2(n+1-j)} R_j^\alpha(x)| \frac{(j+1)^{\gamma+1/2}}{(n+1-j)^{2\gamma-\alpha+3/2}},$$

*where* $N-1 \le 2\gamma-\alpha+1/2 < N \in \mathbf{N}$. *Then there holds* $\|f_n^{\alpha,\gamma}\|_{\infty,\gamma} \le C$ *with* $C$ *independent of* $n$.

## 2 Proof of the Lemma

We use the standard notation $\nu = 4n+2\alpha+2$ and note that on account of formulae (2.5) and (2.7) in [8] there holds for $\alpha > -1$ and some positive $\xi$

$$|e^{-x/2} R_n^\alpha(x)| \le C \begin{cases} 1 & , 0 \le x \le 2/\nu\,, \\ (x\nu)^{-\alpha/2-1/4} & , 1/2\nu \le x \le 3\nu/4\,, \\ (x\nu)^{-\alpha/2}(\nu(\nu^{1/3}+|x-\nu|))^{-1/4} & , \nu/4 \le x \le 2\nu\,, \\ (x\nu)^{-\alpha/2} e^{-\xi x} & , 5\nu/4 \le x\,. \end{cases} \tag{8}$$

Thus, if we choose $n_* := [(\nu-40)/20]$, these estimates are also true for $R_{n-k}^\alpha(x)$ and $R_{n+k+2}^\alpha(x)$ on the $x$-intervals $[0,1/\nu]$, $[1/\nu,\nu/2]$, $[\nu/2,3\nu/2]$, $[3\nu/2,\infty)$, resp., when $k \le n_*$.

We start with the case $N=1$, thus $2\gamma-\alpha < 1/2$ , and decompose $f_n^{\alpha,\gamma}$ as follows

$$f_n^{\alpha,\gamma}(x) = \left(\sum_{k=0}^{n_*} + \sum_{k=n_*+1}^{n}\right) |\Delta_{2(k+1)} R_{n-k}^\alpha(x)| \frac{(n+1-k)^{\gamma+1/2}}{(k+1)^{2\gamma-\alpha+3/2}} =: \Sigma_{n,1}(x) + \Sigma_{n,2}(x)\,. \tag{9}$$

Let us first handle the contribution coming from $\Sigma_{n,2}$. By [8], formula (2.9), one has $\operatorname{ess\,sup}_x |R_n^\alpha(x) x^{\alpha-\gamma} e^{-x/2}| \le C(n+1)^{\gamma-\alpha}$ and, therefore,

$$\operatorname{ess\,sup}_x |\Sigma_{n,2}(x) x^{\alpha-\gamma} e^{-x/2}| \le C(n+1)^{\alpha-2\gamma-3/2} \sum_{k=n_*+1}^{n} (n+1-k)^{\gamma+1/2+(\gamma-\alpha)} \le C \tag{10}$$

uniformly in $n$.

Concerning the estimate of $\Sigma_{n,1}(x)$ we first consider the case $0 < x < 1/\nu$. Use of the identity

$$\Delta_{2k+2} R_{n-k}^\alpha(x) = \frac{x}{\alpha+1}\Big(R_{n-k}^{\alpha+1}(x) + R_{n-k+1}^{\alpha+1}(x) + \cdots + R_{n+k+1}^{\alpha+1}(x)\Big) \tag{11}$$

in combination with the first case of (8) gives

$$|e^{-x/2} \Delta_{2(k+1)} R_{n-k}^\alpha(x)| \le \frac{x e^{-x/2}}{\alpha+1} \sum_{j=0}^{2k+1} |R_{n-k+j}^{\alpha+1}(x)| \le C(k+1)x\,, \tag{12}$$

and thus the desired estimate

$$|x^{\alpha-\gamma}e^{-x/2}\Sigma_{n,1}(x)| \le Cx^{\alpha-\gamma+1}(n+1)^{\gamma+1/2}\sum_{k=0}^{n_*}(k+1)^{\alpha-2\gamma-1/2}$$

$$\le Cx^{\alpha-\gamma+1}(n+1)^{\alpha-\gamma+1} \le C\,.$$

It is similarly simple to deal with the case $x \ge 3\nu/2$. In this case we have by (8) (note $\xi > 0$) that

$$|x^{\alpha-\gamma}e^{-x/2}\Sigma_{n,1}(x)| \le C(n+1)^{\gamma-\alpha/2+1/2}x^{\alpha/2-\gamma}e^{-\xi x}\sum_{k=0}^{n_*}(k+1)^{\alpha-2\gamma-3/2} \le C\,. \qquad (13)$$

To deal with $\Sigma_{n,1}(x)$ for $1/\nu \le x \le \nu/2$ we make use of formula (2.5) in [8] and (11) to obtain

$$|e^{-x/2}\Delta_{2k+2}R^{\alpha}_{n-k}(x)| \le Cx(k+1)(x\nu)^{-\alpha/2-3/4}.$$

If one considers a fixed $x$, $1/\nu \le x \le \nu/2$, one can find a real number $\lambda$, $-1 < \lambda < 1$, such that $\nu^{\lambda}/2 \le x \le \nu^{\lambda}$. Choosing $\mu = 1/2 - \lambda/2 > 0$ we obtain with the previous asymptotic and the second case in (8)

$$|x^{\alpha-\gamma}e^{-x/2}\Sigma_{n,1}(x)| \le C(n+1)^{\gamma+1/2}x^{\alpha-\gamma}e^{-x/2}\sum_{k=0}^{[n_*^{\mu}]}|R^{\alpha}_{n-k}(x)-R^{\alpha}_{n+k+2}(x)|(k+1)^{\alpha-2\gamma-3/2}$$

$$+(n+1)^{\gamma+1/2}x^{\alpha-\gamma}e^{-x/2}\sum_{k=[n_*^{\mu}]+1}^{n_*}(|R^{\alpha}_{n-k}(x)|+|R^{\alpha}_{n+k+2}(x)|)(k+1)^{\alpha-2\gamma-3/2}$$

$$=: \Sigma_{n,3}(x)+\Sigma_{n,4}(x) \le C(n+1)^{\gamma+1/2-(\alpha/2+3/4)}x^{\alpha-\gamma+1-(\alpha/2+3/4)}\sum_{k=0}^{[n_*^{\mu}]}(k+1)^{\alpha-2\gamma-1/2}$$

$$+C(n+1)^{\gamma+1/2-(\alpha/2+1/4)}x^{\alpha-\gamma-(\alpha/2+1/4)}(n+1)^{(\alpha-2\gamma-1/2)\mu} \le C$$

uniformly in $n$.

Concerning an estimate of $\Sigma_{n,1}(x)$ for $x \in [\nu/2, 3\nu/2]$ we may restrict ourselves to $x \in [\nu/2, \nu]$ since the interval $[\nu, 3\nu/2]$ is handled in the same way.

We start with fixed $x$, $\nu - 2\nu^{1/3} \le x \le \nu$, hence $x \approx n+1$, and use the preceding decomposition $|x^{\alpha-\gamma}e^{-x/2}\Sigma_{n,1}(x)| \le \Sigma_{n,3}(x)+\Sigma_{n,4}(x)$ with $\mu = 1/3$. By (8), third case, there follows

$$\Sigma_{n,4}(x) \le C(n+1)^{1/2-1/3}\sum_{k=[n_*^{1/3}]+1}^{n_*}(k+1)^{\alpha-2\gamma-3/2} \le C$$

uniformly in $n$. In order to dominate $\Sigma_{n,3}(x)$ we have to use precise asymptotics for the orthonormal Laguerre functions

$$\mathcal{L}_n^\alpha(x) = \left(\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\right)^{1/2} R_n^\alpha(x) x^{\alpha/2} e^{-x/2}$$

as given in Askey and Wainger [1, p. 699]. We first observe that

$$\Sigma_{n,3}(x) \le C(n+1)^{\alpha/2+1/2} \sum_{k=0}^{[n_*^{1/3}]} \left| \frac{\mathcal{L}_{n-k}^\alpha(x)}{\sqrt{L_{n-k}^\alpha(0)}} - \frac{\mathcal{L}_{n+k+2}^\alpha(x)}{\sqrt{L_{n+k+2}^\alpha(0)}} \right| (k+1)^{\alpha-2\gamma-3/2}$$

$$\le C(n+1)^{1/2} \sum_{k=0}^{[n_*^{1/3}]} |\mathcal{L}_{n-k}^\alpha(x) - \mathcal{L}_{n+k+2}^\alpha(x)| (k+1)^{\alpha-2\gamma-3/2}$$

$$+C(n+1)^{\alpha/2+1/2} \sum_{k=0}^{[n_*^{1/3}]} |\Delta_2 (L_{n-k}^\alpha(0))^{-1/2}| |\mathcal{L}_{n+k+2}^\alpha(x)| (k+1)^{\alpha-2\gamma-3/2}$$

$$=: \Sigma'_{n,3}(x) + \Sigma''_{n,3}(x)$$

(note that $x \approx (n+1)$ ).
That $|\Sigma''_{n,3}(x)| \le C$ holds is obvious when one uses the third case of (8) and observes that for $0 \le k \le n^*$ one has

$$\left| \frac{1}{\sqrt{L_{n-k}^\alpha(0)}} - \frac{1}{\sqrt{L_{n+k+2}^\alpha(0)}} \right| \le \frac{C(k+1)}{(n+1)^{\alpha/2+1}} .$$

The crucial term $\Sigma'_{n,3}(x)$ also turns out to be uniformly bounded when we use the fourth asymptotic in [1, p. 699].

$$\Sigma'_{n,3}(x) \le C(n+1)^{1/2} \sum_{k=0}^{[n_*^{1/3}]} \sum_{j=0}^{k} |\Delta_2 \mathcal{L}_{n+2j-k}^\alpha(x)| (k+1)^{\alpha-2\gamma-3/2}$$

$$\le C(n+1)^{1/2-2/3} \sum_{k=0}^{[n_*^{1/3}]} (k+1)^{\alpha-2\gamma-1/2} \le C.$$

Let us now consider the remaining $x$, $\nu/2 \le x \le \nu - 2\nu^{1/3}$. Then there exists a $\lambda$, $1/3 < \lambda < 1$, such that $\nu - 2\nu^\lambda \le x \le \nu - \nu^\lambda$. Associate to $\lambda$ the number $\mu$, $4\mu = (1-\lambda)/(2\gamma - \alpha + 1/2)$; then obviously $0 < \mu < 1$. Analogously to the above we decompose $\Sigma_{n,1}(x)$ in the following way

$$|x^{\alpha-\gamma} e^{-x/2} \Sigma_{n,1}(x)| \le C(n+1)^{1/2} \sum_{k=0}^{[n_*^\mu]} \sum_{j=0}^{k} |\Delta_2 \mathcal{L}_{n+2j-k}^\alpha(x)| (k+1)^{\alpha-2\gamma-3/2}$$

$$+C(n+1)^{\alpha/2+1/2}\sum_{k=0}^{[n_*^\mu]}|\Delta_2(L_{n-k}^\alpha(0))^{-1/2}||\mathcal{L}_{n+k+2}^\alpha(x)|(k+1)^{\alpha-2\gamma-3/2}$$

$$+C(n+1)^{\alpha+1/2}e^{-x/2}\sum_{k=[n_*^\mu]+1}^{n_*}(|R_{n-k}^\alpha(x)|+|R_{n+k+2}^\alpha(x)|)(k+1)^{\alpha-2\gamma-3/2}$$

$$=:\Sigma_{n,3}'(x)+\Sigma_{n,3}''(x)+\Sigma_{n,4}(x)\,.$$

To estimate $\Sigma_{n,3}'(x)$ we note that by [1, p. 699]

$$|\Delta_2\mathcal{L}_{n+2j-k}^\alpha(x)|\le C(n+2j-k)^{-3/4}|4(n+2j-k)+2\alpha+2-x|^{1/4}\le C(n+1)^{-3/4+\lambda/4}$$

since $n+2j-k\ge n-k\ge n/2$ and $|4(n+2j-k)+2\alpha+2-x|\le C|8j-4k+\nu^\lambda|\le C\nu^\lambda$ (observe that $k\le[n_*^\mu]\le Cn^\lambda$ , $\mu\le\lambda$, and $\lambda>1/3$). Thus

$$\Sigma_{n,3}'(x)\le C(n+1)^{1/2}(n+1)^{-3/4+\lambda/4}\sum_{k=0}^{[n_*^\mu]}(k+1)^{\alpha-2\gamma-1/2}\le C.$$

Analogously we have that $\Sigma_{n,3}''(x)$ is uniformly bounded in $n$ on the interval $[\nu-2\nu^\lambda,\nu-\nu^\lambda]$. To dominate $\Sigma_{n,4}(x)$ we note that by (8), third case, the worst contribution estimate comes from $|R_{n-k}^\alpha(x)|$ so that the $k$-range has to be examined in order to know which asymptotic to use. Since $|4(n-k)+2\alpha+2-x|\approx|\nu^\lambda-4k|$ we further split up

$$\Sigma_{n,4}(x)\le C(n+1)^{\alpha+1/2}e^{-x/2}\sum_{k=[n_*^\mu]+1}^{n_*}\ldots=:\Sigma_{n,4}'(x)+\Sigma_{n,4}''(x)$$

where in $\Sigma'$ only over those $k$ is summed for which $|\nu^\lambda-4k|\ge\nu^\lambda/2$, thus the summation variable $k$ in the sum associated to $\Sigma''$ runs from $[\nu^\lambda/8]$ to $[3\nu^\lambda/8]$. Then, dealing with $\Sigma'$ and observing that for these $k$ there holds $|x^{\alpha/2}e^{-x/2}R_{n-k}^\alpha(x)|\le C(n+1)^{-1/4-\alpha/2}\nu^{-\lambda/4}$ we obtain

$$\Sigma_{n,4}'(x)\le C(n+1)^{1/4}\nu^{-\lambda/4}\sum_{k=[n_*^\mu]+1}^{n_*}(k+1)^{\alpha-2\gamma-3/2}\le C$$

by the choice of $\lambda$ and $\mu$ since $\alpha\le2\gamma$ by hypothesis. Turning to $\Sigma''$ we note that for these $k$ in any case $|x^{\alpha/2}e^{-x/2}R_{n-k}^\alpha(x)|\le C(n+1)^{-1/3-\alpha/2}$ holds so that

$$\Sigma_{n,4}''(x)\le C(n+1)^{1/2}(n+1)^{-1/3}\sum_{k=[\nu^\lambda/8]}^{[3\nu^\lambda/8]}(k+1)^{\alpha-2\gamma-3/2}\le C$$

since $\lambda>1/3$.

Now let $\alpha$, $\gamma$ be such that $N-1 \leq 2\gamma - \alpha + 1/2 < N$, $N \in \mathbf{N}$, $N \geq 2$. We make again a decomposition analogous to (9), this time choosing $n_* := [(\nu - 40)/100N]$, replacing $\Delta$ by $\Delta^N$, and denoting the two resulting sums by $\Sigma_{n,1,N}(x)$ and $\Sigma_{n,2,N}(x)$. The estimate analogous to (10) remains valid on account of the triangle inequality. If instead of (12) one uses the estimate $|\Delta^N_{2(k+1)} R^\alpha_{n-k}(x)| \leq C(k+1)^N x^N$ it is clear that $|x^{\alpha-\gamma} e^{-x/2} \Sigma_{n,1,N}(x)| \leq C$ holds for $0 < x < 1/\nu$. Also the analog of (13) is obvious for $x \geq 3\nu/2$ so that there is only to discuss the case $1/\nu \leq x \leq 3\nu/2$.

By definition we have $\Delta^N_{2k+2} = \Delta_{2k+2}\Delta^{N-1}_{2k+2}$. Following Kal'neĭ [6] we use the equality

$$\Delta^{N-1}_{2k+2} R^\alpha_{n-k}(x) = \sum_{m=0}^{k} \cdots \sum_{l=0}^{k} \Delta^{N-1}_2 R^\alpha_{n-k+2l+\cdots+2m}(x) \tag{14}$$

with $(N-1)$ summations, and also observe that, by formula (3) in Part I of [2],

$$\Delta^{N-1}_2 R^\alpha_k(x) = \sum_{j=0}^{N-1} C_{j,N} \Delta^{N-1}_1 R^\alpha_{k+j}(x) = C_{\alpha,N} \sum_{j=0}^{N-1} C_{j,N} x^{N-1} R^{\alpha+N-1}_{k+j}(x)\,.$$

Hence, when we need to work with $\Delta^N_{2k+2} R^\alpha_{n-k}(x)$, $0 \leq k \leq n_* = [(\nu-40)/100N]$, it suffices to replace this by a linear combination (in $j$, $0 \leq j \leq N-1$), of $(k+1)^{N-1}$ terms (in $i$ coming from (14)) of the type

$$x^{N-1}\Delta_{2k+2} R^{\alpha+N-1}_{n+j+2i-k}(x)\,,\ 0 \leq j \leq N-1\,,\ 0 \leq i \leq k(N-1)\,,\ 0 \leq k \leq n_*\,.$$

We note, since $n_*$ is chosen so small, that on $1/\nu \leq x \leq 3\nu/2$ the $R^\alpha_{n-k+2i}$ have the same asymptotics for the relevant $k, i$ as well as $R^{\alpha+N-1}_{n+j+2i-k}$ for the relevant $k, j, i$.

Let us now consider $\Sigma_{n,1,N}(x)$ on $1/\nu \leq x \leq \nu/2$. As in the case $N = 1$ we fix $x$, can find a real $\lambda$, $-1 < \lambda < 1$, such that $\nu^\lambda/2 \leq x \leq \nu^\lambda$ and choose $\mu = 1/2 - \lambda/2 > 0$. Then, by the preceding discussion and with the abbreviation $N^*_k := (2k+1)(N-1)$,

$$|x^{\alpha-\gamma} e^{-x/2} \Sigma_{n,1,N}(x)| \leq C(n+1)^{\gamma+1/2} x^{\alpha-\gamma} e^{-x/2} \sum_{k=0}^{[n_*^\mu]} |\Delta^N_{2k+2} R^\alpha_{n-k}(x)| (k+1)^{\alpha-2\gamma-3/2}$$

$$+C(n+1)^{\gamma+1/2} x^{\alpha-\gamma} e^{-x/2} \sum_{k=[n_*^\mu]+1}^{n_*} \sup_{0\leq j\leq N} |R^\alpha_{n+2j(k+1)-k}(x)| (k+1)^{\alpha-2\gamma-3/2}$$

$$\leq C(n+1)^{\gamma+1/2} x^{\alpha-\gamma+(N-1)} e^{-x/2} \sum_{k=0}^{[n_*^\mu]} \sup_{0\leq j\leq N^*_k} |\Delta_{2k+2} R^{\alpha+N-1}_{n+j-k}(x)| (k+1)^{\alpha-2\gamma-3/2+(N-1)}$$

$$+C(n+1)^{\gamma+1/2-(\alpha/2+1/4)} x^{\alpha-\gamma-(\alpha/2+1/4)} \sum_{k=[n_*^\mu]+1}^{n_*} \sup_{0\leq j\leq N} (k+1)^{\alpha-2\gamma-3/2}$$

$$\leq C(n+1)^{\gamma-\alpha/2-N/2+1/4}x^{\alpha/2-\gamma+N/2-1/4}\sum_{k=0}^{[n_*^\mu]}(k+1)^{\alpha-2\gamma-3/2+N}+C\leq C$$

uniformly in $n$.

To complete the proof of Lemma we may restrict ourselves, as in the case $N=1$, to discussing the $x$-interval $[\nu/2,\nu]$. We start with $\nu-2\nu^{1/3}\leq x\leq\nu$ and set $6\mu:=1/(2\gamma-\alpha+1/2)$. Then, as in the case $1/\nu\leq x\leq\nu/2$,

$$|x^{\alpha-\gamma}e^{-x/2}\Sigma_{n,1,N}(x)|$$

$$\leq C(n+1)^{\alpha+N-1/2}e^{-x/2}\sum_{k=0}^{[n_*^\mu]}\sup_{0\leq j\leq N_k^*}|\Delta_{2k+2}R_{n+j-k}^{\alpha+N-1}(x)|(k+1)^{\alpha-2\gamma-3/2+(N-1)}$$

$$+C(n+1)^{\alpha+1/2}e^{-x/2}\sum_{k=[n_*^\mu]+1}^{n_*}\sup_{0\leq j\leq N}|R_{n+2j(k+1)-k}^{\alpha}(x)|(k+1)^{\alpha-2\gamma-3/2}$$

$$=:\Sigma_{n,3,N}(x)+\Sigma_{n,4,N}(x)$$

That $\Sigma_{n,4,N}$ is uniformly bounded follows by our choice of $\mu$ when we use the third line of (8). Now

$$\Sigma_{n,3,N}(x)\leq C(n+1)^{(\alpha+N)/2}\sum_{k=0}^{[n_*^\mu]}\sup_{0\leq j\leq N_k^*}\left|\Delta_{2k+2}\frac{\mathcal{L}_{n+j-k}^{\alpha+N-1}(x)}{\sqrt{L_{n+j-k}^{\alpha+N-1}(0)}}\right|(k+1)^{\alpha-2\gamma-5/2+N}$$

$$\leq C(n+1)^{1/2}\sum_{k=0}^{[n_*^\mu]}\sup_{0\leq j\leq N_k^*}|\Delta_{2k+2}\mathcal{L}_{n+j-k}^{\alpha+N-1}(x)|(k+1)^{\alpha-2\gamma-5/2+N}$$

$$+C(n+1)^{(\alpha+N)/2}\sum_{k=0}^{[n_*^\mu]}\sup_{0\leq j\leq N_k^*}|\Delta_{2k+2}(L_{n+j-k}^{\alpha+N-1}(0))^{-1/2}||\mathcal{L}_{n+j+k+2}^{\alpha+N-1}(x)|(k+1)^{\alpha-2\gamma-5/2+N}$$

$$\leq C(n+1)^{-1/6}\sum_{k=0}^{[n_*^\mu]}(k+1)^{\alpha-2\gamma-3/2+N}+C(n+1)^{-1/2-1/3}\sum_{k=0}^{[n_*^\mu]}(k+1)^{\alpha-2\gamma-3/2+N}\leq C$$

uniformly in $n$ since $N\geq 2$. Hence there remains to consider fixed $x\in[\nu/2,\nu-2\nu^{1/3}]$. As in the $(N=1)$-case there is a $\lambda$, $1/3<\lambda<1$, such that $\nu-2\nu^\lambda\leq x\leq\nu-\nu^\lambda$; choose $\mu$, $4\mu=(1-\lambda)/(2\gamma-\alpha+1/2)$. Since $1\leq N-1\leq 2\gamma-\alpha+1/2$ we obviously have $\mu<1/6$. Make the same decomposition as in the preceding $[\nu-2\nu^{1/3},\nu]$-case. Concerning the $\sum_{k=0}^{[n_*^\mu]}$-contribution we use the third line of the asymptotics in [1, p. 699] and observe that $|x-4(n+j-k)-2\alpha-2|\leq C\nu^\lambda$ since $0\leq j\leq(2k+1)(N-1)\leq Cn^\mu$, thus

$$\Sigma_{n,3,N}(x)\leq C(n+1)^{1/2}\sum_{k=0}^{[n_*^\mu]}\sup_{0\leq j\leq N_k^*}\sum_{i=0}^{k}|\Delta_2\mathcal{L}_{n+j+2i-k}^{\alpha+N-1}(x)|(k+1)^{\alpha-2\gamma-5/2+N}$$

$$+C(n+1)^{-1/2}\sum_{k=0}^{[n_*^\mu]}\sup_{0\le j\le N_k^*}|\mathcal{L}_{n+j+k+2}^{\alpha+N-1}(x)|(k+1)^{\alpha-2\gamma-3/2+N}$$

$$\le C(n+1)^{1/2-3/4+\lambda/4}\sum_{k=0}^{[n_*^\mu]}(k+1)^{\alpha-2\gamma-3/2+N}$$

$$+C(n+1)^{-1/2-1/4-\lambda/4}\sum_{k=0}^{[n_*^\mu]}(k+1)^{\alpha-2\gamma-3/2+N}\le C\,.$$

In order to dominate the $\sum_{k=[n_*^\mu]}^{n_*}$-contribution we use the method, analogous to the corresponding ($N=1$)-case. Hence, we split $\Sigma_{n,4,N}$ into a sum $\Sigma'$ where the summation variable $k$ also satisfies the inequality $|\nu^\lambda-4k|\ge\nu^\lambda/2$ and a sum $\Sigma''$ over the remaining $k$'s. Then, as in the corresponding ($N=1$)-case, both contributions turn out to be uniformly bounded.

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