On a restriction problem of de Leeuw type for Laguerre multipliers

George Gasper[1] and Walter Trebels[2]

Dedicated to Károly Tandori on the occasion of his 70th birthday

(Published in Acta Math. Hungar. 68(1–2)(1995), 135–149)

Abstract. In 1965 K. de Leeuw [3] proved among other things in the Fourier transform setting: *If a continuous function* $m(\xi_1,\dots,\xi_n)$ *on* $\mathbf{R}^n$ *generates a bounded transformation on* $L^p(\mathbf{R}^n)$, $1 \le p \le \infty$, *then its trace* $\tilde{m}(\xi_1,\dots,\xi_k) = m(\xi_1,\dots,\xi_k,0,\dots,0)$, $k<n$, *generates a bounded transformation on* $L^p(\mathbf{R}^k)$. In this paper, the analogous problem is discussed in the setting of Laguerre expansions of different orders.

Key words. Laguerre polynomials, embeddings of multiplier spaces, projections, transplantation, weighted Lebesgue spaces

AMS(MOS) subject classifications. 33C45, 42A45, 42C10

1 Introduction

The purpose of this paper is to discuss the question: suppose $\{m_k\}_{k\in\mathbf{N}_0}$ generates a bounded transformation with respect to a Laguerre function expansion of order α on some L^p–space, does it also generate a corresponding bounded map with respect to a Laguerre function expansion of order β? To become more precise let us first introduce some notation. Consider the Lebesgue spaces

$$L^p_{w(\gamma)} = \{f:\ \|f\|_{L^p_{w(\gamma)}} = (\int_0^\infty |f(x)e^{-x/2}|^p x^\gamma\,dx)^{1/p} < \infty\}\ , \quad 1\le p<\infty,$$

$$L^\infty_{w(\gamma)} = \{f:\ \|f\|_{L^\infty_{w(\gamma)}} = \text{ess sup}_{x>0}|f(x)e^{-x/2}| < \infty\}, \qquad p=\infty,$$

where $\gamma > -1$. Let $L_n^\alpha(x)$, $\alpha>-1$, $n\in\mathbf{N}_0$, denote the classical Laguerre polynomials (see Szegö [15, p. 100]) and set

$$R_n^\alpha(x) = L_n^\alpha(x)/L_n^\alpha(0), \qquad L_n^\alpha(0) = A_n^\alpha = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

[1]Department of Mathematics, Northwestern University, Evanston, IL 60208, USA. The work of this author was supported in part by the National Science Foundation under grant DMS-9103177.

[2]Fachbereich Mathematik, TH Darmstadt, Schloßgartenstr.7, D–64289 Darmstadt, Germany.

Associate to f its formal Laguerre series

$$f(x) \sim (\Gamma(\alpha+1))^{-1} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\alpha(x),$$

where the Fourier Laguerre coefficients of f are defined by

$$\hat{f}_\alpha(n) = \int_0^\infty f(x) R_n^\alpha(x) x^\alpha e^{-x}\, dx \tag{1}$$

(if the integrals exist). A sequence $m = \{m_k\}_{k\in\mathbf{N}_0}$ is called a (bounded) multiplier from $L^p_{w(\gamma)}$ into $L^q_{w(\delta)}$, notation $m \in M^{p,q}_{\alpha;\gamma,\delta}$, if

$$\| \sum_{k=0}^{\infty} m_k \hat{f}_\alpha(k) L_k^\alpha \|_{L^q_{w(\delta)}} \le C \| \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L_k^\alpha \|_{L^p_{w(\gamma)}}$$

for all polynomials f; the smallest constant C for which this holds is called the multiplier norm $\|m\|_{M^{p,q}_{\alpha;\gamma,\delta}}$. For the sake of simplicity we write $M^{p,q}_{\alpha;\gamma} := M^{p,q}_{\alpha;\gamma,\gamma}$ if $\gamma = \delta$ and, if additionally $p = q$, $M^p_{\alpha;\gamma} := M^{p,p}_{\alpha;\gamma}$.

We are **mainly** interested in the question: when is $M^{p,q}_{\alpha;\alpha}$ continuously embedded in $M^{p,q}_{\beta;\beta}$:

$$M^{p,q}_{\alpha;\alpha} \hookrightarrow M^{p,q}_{\beta;\beta}, \quad 1 \le p \le q \le \infty\ ?$$

The Plancherel theory immediately yields

$$l^\infty = M^2_{\alpha;\alpha} = M^2_{\beta;\beta}, \quad \alpha, \beta > -1.$$

A combination of sufficient multiplier conditions with necessary ones indicates which results are to be expected. To this end, define the fractional difference operator Δ^δ of order δ by

$$\Delta^\delta m_k = \sum_{j=0}^{\infty} A_j^{-\delta-1} m_{k+j}$$

(whenever the series converges), the classes $wbv_{q,\delta}$, $1 \le q \le \infty$, $\delta > 0$, of weak bounded variation (see [5]) of bounded sequences which have finite norm $\|m\|_{q,\delta}$, where

$$\begin{aligned} \|m\|_{q,\delta} &:= \sup_k |m_k| + \sup_{N\in\mathbf{N}_0} \left(\sum_{k=N}^{2N} |(k+1)^\delta \Delta^\delta m_k|^q \frac{1}{k+1} \right)^{1/q}, \quad && q < \infty, \\ \|m\|_{\infty,\delta} &:= \sup_k |m_k| + \sup_{N\in\mathbf{N}_0} |(k+1)^\delta \Delta^\delta m_k| , && q = \infty. \end{aligned}$$

Observing the duality (see [14])

$$M^p_{\alpha;\gamma} = M^{p'}_{\alpha;\alpha p' - \gamma p'/p}, \quad -1 < \gamma < p(\alpha+1) - 1, \quad 1 < p < \infty, \tag{2}$$

where $1/p+1/p'=1$, we may restrict ourselves to the case $1<p<2$. The Corollary 1.2 b) in [14] gives the embedding

$$M^p_{\alpha;\alpha} \hookrightarrow wbv_{p',s}, \quad s=(2\alpha+2/3)(1/p-1/2), \quad \alpha>-1/3, \tag{3}$$

when $(2\alpha+2)(1/p-1/2)>1/2$. Theorem 5 in [5] gives the first embedding in

$$wbv_{p',s} \hookrightarrow wbv_{2,s} \hookrightarrow M^p_{\beta;\beta},$$

whereas the last one follows from Corollaries 1.2 and 4.5 in [14] provided $s>\max\{(2\beta+2)(1/p-1/2),\ 1\}$, $\beta>-1$. Hence, choosing $\gamma=\alpha$ in (2), we obtain

Proposition 1.1 *Let $1<p<\infty$ and α be such that $(2\alpha+2/3)|1/p-1/2|>1$. Then*

$$M^p_{\alpha;\alpha} \hookrightarrow M^p_{\beta;\beta}\,, \quad -1<\beta<\alpha-2/3.$$

In the same way we can derive a result for $M^{p,q}$–multipliers. The necessary condition in [6, Cor. 1.3] can easily be extended in the sense of [6, Cor. 2.5 b)] to

$$\sup_k |(k+1)^\sigma m_k| + \sup_n \Big(\sum_{k=n}^{2n} |(k+1)^{\sigma+s}\Delta^s m_k|^{q'}/k\Big)^{1/q'} \le C\|m\|_{M^{p,q}_{\alpha;\alpha}}\,,$$

where $\alpha>-1/3$, $1/q=1/p-\sigma/(\alpha+1)$, $1<p<q<2$, $(\alpha+1)(1/q-1/2)>1/4$, and $s=(2\alpha+2/3)(1/q-1/2)>0$. Using this and the sufficient condition for $M^{p,q}_{\beta;\beta}$–multipliers given in [4, Cor. 1.2], which is proved only for $\beta\ge 0$, we obtain

$$M^{p,q}_{\alpha;\alpha} \hookrightarrow M^{p,q}_{\beta;\beta}\,, \quad 0\le\beta<\alpha-2/3,\ (2\alpha+2/3)(1/q-1/2)>1,\ 1<p<q<2.$$

In this context let us mention that the same technique yields for $1<p,q<2$

$$M^p_{\alpha;\alpha} \hookrightarrow M^q_{\beta;\beta}\,, \quad (2\alpha+2/3)(1/p-1/2)>\max\{(2\beta+2)(1/q-1/2),\ 1\}. \tag{4}$$

This embedding is in so far interesting as it allows to go from p, $1<p<2$, to $q\ne p$, $1<q<2$, connected with a loss in the size of β if $q<p$ or a gain in β if $1<p<q<2$; e.g.

$$M^{4/3}_{10;10} \hookrightarrow M^q_{5;5}\,, \quad 1.08\le q\le 2, \quad \text{or} \quad M^{8/7}_{2;2} \hookrightarrow M^q_{4;4}\,, \quad 3/2\le q\le 2.$$

Improvements of (4) can be expected by better necessary conditions and/or better sufficient conditions; but this technique **cannot** give something like

$$M^p_{\alpha;\alpha} \hookrightarrow M^q_{\beta;\beta}, \quad (\alpha+1)(1/p-1/2)>(\beta+1)(1/q-1/2),\ 1<p,\ q<2,$$

which is suggested by (4) when choosing "large" α with p near 2 since then the number $4(1/p-1/2)/3$, which describes the smoothness gap between the necessary conditions and the sufficient conditions in [14, Cor. 1.2], is "negligible".

Concerning the general problem "When does $M^{p,q}_{\alpha;\gamma_1,\delta_1} \hookrightarrow M^{p,q}_{\beta;\gamma_2,\delta_2}$ hold?",
we mention results in Stempak and Trebels [14, Cor. 4.3]: For $1<p<\infty$ there holds

$$M^p_{\beta;\beta p/2+\delta} = M^p_{0;\delta} \quad \text{if} \begin{cases} -1-\beta p/2<\delta<p-1+\beta p/2\,, & -1<\beta<0\,, \\ -1<\delta<p-1\,, & 0\le\beta, \end{cases}$$

which for $\delta=0$ contains half of Kanjin's [9] result and for $\delta=p/4-1/2$ Thangavelu's [16]. In particular, there holds for $-1<\beta<\alpha$, $1<p<\infty$,

$$M^p_{\beta;\beta} = M^p_{\beta;\,\beta p/2+\beta p(1/p-1/2)} = M^p_{\alpha;\,\alpha p/2+\beta p(1/p-1/2)}\,, \quad (2\beta+2)|1/p-1/2|<1. \qquad (5)$$

These results are based on Kanjin's [9] transplantation theorem and its weighted version in [14]. The latter gives further insight into our problem in so far as it implies that the restriction $\beta<\alpha-2/3$ in Proposition 1.1 is not sharp.
To this end we first note that the following extension of Corollary 4.4 in [14] holds

$$wbv_{2,s} \hookrightarrow M^p_{\alpha;\,\alpha p/2+\eta(p/2-1)}, \quad 0\le\eta\le1, \quad 1<p\le2, \quad s>1/p.$$

(For the proof observe that for $\alpha=0$ the parameter $\gamma=\eta(p/2-1)$, $0\le\eta\le1$, is admissible in [14, Theorem 1.1] and then follow the argumentation of [14, Cor. 4.4].) This combined with (3) yields for $s=(2\alpha+2/3)(1/p-1/2)>1/p$

$$M^p_{\alpha;\,\alpha} \hookrightarrow wbv_{2,s} \hookrightarrow M^p_{\alpha;\,\alpha p/2+p/2-1}\,, \quad 1<p\le2, \quad \alpha>(p+1)/(6-3p).$$

Thus, by interpolation with change of measure,

$$M^p_{\alpha;\,\alpha} \hookrightarrow M^p_{\alpha;\,\alpha p/2+\delta}\,, \quad p/2-1\le\delta\le\alpha-\alpha p/2, \quad \alpha>(p+1)/(6-3p).$$

Since (5) gives

$$M^p_{\alpha;\,\alpha p/2+\beta p(1/p-1/2)} = M^p_{\beta;\,\beta}$$

we arrive at

Proposition 1.2 *Let $1<p\le2$ and $\alpha>(p+1)/(6-3p)$. Then*

$$M^p_{\alpha;\,\alpha} \hookrightarrow M^p_{\beta;\,\beta}\,, \quad (2\beta+2)(1/p-1/2)<1, \quad -1<\beta<\alpha.$$

The first restriction on β is equivalent to $\beta<(2p-2)/(2-p)$. This combined with the restriction on α gives $\alpha-\beta>(7-5p)/(6-3p)$, the latter being decreasing in p and taking the value $2/3$ at $p=1$. Hence Proposition 1.2 is an improvement of the previous one for all $1<p<2$ provided $(p+1)/(6-3p)<\alpha\le(2p-2)/(2-p)$.

For big α's, Proposition 1.1 is certainly better. If in the transplantation theorem in [14] higher exponents could be allowed in the power weight – which is possible in the Jacobi expansion case as shown by Muckenhoupt [12] – the technique just used would give the embedding when $-1<\beta<\alpha$, $1<p<2$, and $\alpha>(p+1)/(6-3p)$. Summarizing, it is reasonable to

conjecture $$M^{p,q}_{\alpha;\alpha} \hookrightarrow M^{p,q}_{\beta;\beta}, \quad -1<\beta<\alpha, \quad 1\le p\le q\le\infty.$$

Apart from the above fragmentary results, so far we can only prove the conjecture in the extreme case when $q=\infty$ and $\beta\ge 0$; the latter restriction arises from the fact that we have to make use of the twisted Laguerre convolution (see [7]) which is proved till now only for Laguerre polynomials $L^\alpha_n(x)$ with $\alpha\ge 0$. Our main result is

Theorem 1.3 *If $1\le p\le\infty$, then*

$$M^{p,\infty}_{\alpha;\alpha} \hookrightarrow M^{p,\infty}_{\beta;\beta}, \quad 0\le\beta<\alpha.$$

Remarks. 1) One could speculate that an interpolation argument applied to

$$M^2_{\alpha;\alpha}=M^2_{\beta;\beta}, \quad M^\infty_{\alpha;\alpha}=M^1_{\alpha;\alpha}\hookrightarrow M^1_{\beta;\beta}=M^\infty_{\beta;\beta}, \quad \beta<\alpha,$$

could give the open case $M^p_{\alpha;\alpha}\hookrightarrow M^p_{\beta;\beta}$, $1<p<2$. In this respect we mention a result of Zafran [17, p. 1412] for the Fourier transform pointed out to us by A. Seeger:

> *Denote by $M^p(\mathbf{R})$ the set of bounded Fourier multipliers on $L^p(\mathbf{R})$ and by $M^\wedge(\mathbf{R})$ the set of Fourier transforms of bounded measures on $\mathbf{R}$. Then $M^p(\mathbf{R})$, $1<p<2$, is* **not** *an interpolation space with respect to the pair $(M^\wedge(\mathbf{R}), L^\infty(\mathbf{R}))$.*

Thus de Leeuw's result mentioned at the beginning cannot be proved by interpolation.

2) It is perhaps amazing to note that the wbv–classes do not play only an auxiliary role in dealing with the above formulated general problem. In the framework of one-dimensional Fourier transforms/series this was shown by Muckenhoupt, Wheeden, and Wo-Sang Young [13]. That this phenomenon also occurs in the framework of Laguerre expansions can be seen from the following two theorems.

Theorem 1.4 *If $\alpha>-1$, $\alpha\ne 0$, then*

$$wbv_{2,1}\hookrightarrow M^2_{\alpha;\alpha+1}.$$

In the case $-1<\alpha<0$ the multiplier operator is defined only on the subspace $\{f\in L^2_{w(\alpha+1)} : \hat f_\alpha(0)=0\}$.

Theorem 1.5 *If $\alpha > -1$, then*

$$M^2_{\alpha;\,\alpha+1} \hookrightarrow wbv_{2,1}\,.$$

A combination of these two results leads to

$$M^2_{\alpha;\,\alpha+1} = M^2_{\beta;\,\beta+1} = wbv_{2,1}\,, \quad \alpha,\beta > -1, \quad \alpha,\beta \neq 0, \tag{6}$$

and a combination with [14, (19)] gives

$$M^2_{\alpha;\,\alpha+1} \hookrightarrow M^p_{\alpha;\,\alpha}\,, \quad \alpha \geq 0, \quad (2\alpha+2)/(\alpha+1) < p \leq 2.$$

2 Proof of Theorem 1.3

Theorem 1.3 is an immediate consequence of the combination of the following two theorems.

Theorem 2.1 *Let $f \in L^p_{w(\alpha)}$ with $\alpha > -1$ when $1 \leq p < \infty$ and $\alpha \geq 0$ when $p = \infty$. Then there exists a function $g \in L^p_{w(\beta)}$, $-1 < \beta < \alpha$, with*

$$g(x) \sim (\Gamma(\beta+1))^{-1} \sum_{k=0}^{\infty} \hat{f}_\alpha(k) L^\beta_k(x), \quad \|g\|_{L^p_{w(\beta)}} \leq C\|f\|_{L^p_{w(\alpha)}}\,.$$

Proof

First let $1 \leq p < \infty$ and, without loss of generality, let f be a polynomial (these are dense in $L^p_{w(\alpha)}$). We recall the projection formula (3.31) in Askey and Fitch [2]

$$e^{-x} L^\beta_n(x) = \frac{1}{\Gamma(\alpha-\beta)} \int_x^\infty (y-x)^{\alpha-\beta-1} e^{-y} L^\alpha_n(y)\,dy, \quad -1 < \beta < \alpha.$$

Then the following computations are justified.

$$\begin{aligned}
\|g\|_{L^p_{w(\beta)}} &= C\left(\int_0^\infty |\sum_{k=0}^\infty \hat{f}_\alpha(k) L^\beta_k(x) e^{-x/2}|^p x^\beta dx\right)^{1/p} \\
&= C\left(\int_0^\infty \left|\int_x^\infty (y-x)^{\alpha-\beta-1} e^{-y} \sum_{k=0}^\infty \hat{f}_\alpha(k) L^\alpha_k(y)\,dy\right|^p x^\beta e^{xp/2} dx\right)^{1/p} \\
&\leq C\int_1^\infty (t-1)^{\alpha-\beta-1} \left(\int_0^\infty |\sum_k \hat{f}_\alpha(k) L^\alpha_k(xt) x^{\alpha-\beta+\beta/p} e^{-x(t-1/2)}|^p dx\right)^{1/p} dt
\end{aligned}$$

after a substitution and application of the integral Minkowski inequality. Additional substitutions lead to

$$\begin{aligned}\|g\|_{L^p_{w(\beta)}} &\le C\int_0^\infty s^{\alpha-\beta-1}(s+1)^{\beta/p'-\alpha-1/p}\times\\ &\qquad\left(\int_0^\infty |\sum_k \hat f_\alpha(k)L_k^\alpha(y)e^{-y/2}y^{(\alpha-\beta)/p'}e^{-ys/2(s+1)}|^p y^\alpha\,dy\right)^{1/p} ds\\ &\le C\int_0^\infty s^{(\alpha-\beta)/p-1}(s+1)^{-(\alpha+1)/p}\left(\int_0^\infty |\sum_k \hat f_\alpha(k)L_k^\alpha(y)e^{-y/2}|^p y^\alpha\,dy\right)^{1/p} ds,\end{aligned}$$

where we used the inequality $y^{(\alpha-\beta)/p'}e^{-ys/2(s+1)} \le C((s+1)/s)^{(\alpha-\beta)/p'}$. Since $-1 < \beta < \alpha$ it is easily seen that the outer integration only gives a bounded contribution. If $f \in L^\infty_{w(\alpha)}$ then $|(k+1)^{-1/2}\hat f_\alpha(k)| \le C\|f\|_{L^\infty_{w(\alpha)}}$ by [10, Lemma 1] and, therefore, the Abel-Poisson means of an arbitrary $f \in L^\infty_{w(\alpha)}$ can be represented by

$$P_r f(x) = (\Gamma(\alpha+1))^{-1}\sum_k r^k \hat f_\alpha(k)L_k^\alpha(x), \quad 0 \le r < 1, \quad x \ge 0,$$

and, by the convolution theorem in Görlich and Markett [7, p. 169],

$$\|P_r f\|_{L^\infty_{w(\alpha)}} \le C\|f\|_{L^\infty_{w(\alpha)}}, \quad 0 \le r < 1, \quad \alpha \ge 0.$$

A slight modification of the argument in the case $1 \le p < \infty$ shows that

$$\|g_r\|_{L^\infty_{w(\beta)}} := \|(\Gamma(\beta+1))^{-1}\sum_k r^k \hat f_\alpha(k)L_k^\beta\|_{L^\infty_{w(\beta)}} \le C\|P_r f\|_{L^\infty_{w(\alpha)}} \le C\|f\|_{L^\infty_{w(\alpha)}}.$$

By the weak* compactness there exists a function $g \in L^\infty_{w(\beta)}$ with $\hat g_\beta(k) = \hat f_\alpha(k)$ and $\|g\|_{L^\infty_{w(\beta)}} \le \liminf_{k\to\infty}\|g_{r_k}\|_{L^\infty_{w(\beta)}}$ for a suitable sequence $r_k \to 1^-$; hence also the assertion in the case $p = \infty$.

Theorem 2.2 *For $\alpha \ge 0$ there holds*

i) $\qquad M^{1,p}_{\alpha;\alpha} = M^{p',\infty}_{\alpha;\alpha} = (L^p_{w(\alpha)})\widehat{}, \quad 1 < p \le \infty,$

ii) $\qquad M^{1,1}_{\alpha;\alpha} = M^{\infty,\infty}_{\alpha;\alpha} = \{m = \{m_k\}_{k\in\mathbf{N}_0} : \|P_r(m)\|_{L^1_{w(\alpha)}} = O(1),\ r \to 1^-\},$

where $P_r(m)(x) = (\Gamma(\alpha+1))^{-1}\sum_k r^k m_k L_k^\alpha(x)$.

Proof

The first equalities in *i)* and *ii)* are the standard duality statements. Let us briefly indicate the second equalities (which are also more or less standard).

If $m = \{m_k\}_{k\in\mathbf{N}_0}$ are the Fourier Laguerre coefficients of an $L^p_{w(\alpha)}$–function, $1 < p \le \infty$, or in the case $p = 1$ of a bounded measure with respect to the weight $e^{-x/2}x^\alpha$, then

Young's inequality in Görlich and Markett [7] (or a slight extension of it to measures in the case $p = 1$) shows that $m \in M^{p',\infty}_{\alpha;\alpha}$.
Conversely, associate formally to a sequence $m = \{m_k\}$ an operator T_m by

$$T_m f(x) \sim (\Gamma(\alpha+1))^{-1} \sum_{k=0}^{\infty} m_k \hat{f}_\alpha(k) L_k^\alpha(x). \tag{7}$$

Then, in essentially the notation of Görlich and Markett [7],

$$T_m(P_r f)(x) = P_r(m) * f(x) = \int_0^\infty T_x^\alpha(P_r(m)(y)) f(y) e^{-y} y^\alpha dy,$$

where T_x^α is the Laguerre translation operator. If $\|f\|_{L^{p'}_{w(\alpha)}} = 1$ then

$$\|T_m(P_r f)\|_{L^\infty_{w(\alpha)}} \le \|m\|_{M^{p',\infty}_{\alpha;\alpha}} \|P_r f\|_{L^{p'}_{w(\alpha)}} \le C\|m\|_{M^{p',\infty}_{\alpha;\alpha}},$$

and hence, by the converse of Hölder's inequality,

$$\sup_{\|f\|_{L^{p'}_{w(\alpha)}}=1} \left| \int_0^\infty T_x^\alpha(P_r(m)(y)) e^{-y/2} y^{\alpha/p} f(y) e^{-y/2} y^{\alpha/p'} dy \right|$$

$$= \|T_x^\alpha(P_r(m))\|_{L^p_{w(\alpha)}} \le C\|m\|_{M^{p',\infty}_{\alpha;\alpha}}$$

for $x \ge 0,\ 0 \le r < 1$. In particular, for $x = 0$ we obtain

$$\|P_r(m)\|_{L^p_{w(\alpha)}} \le C\|m\|_{M^{p',\infty}_{\alpha;\alpha}}, \quad 0 \le r < 1.$$

Now weak* compactness gives the desired converse embedding.

3 Proof of Theorems 1.4 and 1.5

The proof relies heavily on the Parseval formula

$$\frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{\infty} A_k^\alpha |\hat{f}_\alpha(k)|^2 = \int_0^\infty |f(x)e^{-x/2}|^2 x^\alpha dx \tag{8}$$

and its extension

$$\sum_{k=0}^{\infty} A_k^{\alpha+\lambda} |\Delta^\lambda \hat{f}_\alpha(k)|^2 \approx \int_0^\infty |f(x)e^{-x/2}|^2 x^{\alpha+\lambda} dx, \quad \lambda \ge 0, \tag{9}$$

which is a consequence of the formula

$$\Delta^\lambda \hat{f}_\alpha(k) = C_{\alpha,\lambda} \hat{f}_{\alpha+\lambda}(k) \tag{10}$$

(see e.g. the proof of Lemma 2.1 in [6]). For the proof of Theorem 1.4 we further need the following discrete analog of the $p = 2$ case of a weighted Hardy inequality in Muckenhoupt [11] whose proof can at once be read off from [11] by replacing the integrals there by sums and using the fact that

$$a \leq 2(a+b)^{1/2}[(a+b)^{1/2} - b^{1/2}]$$

when $a,\, b \geq 0$; also see the extensions in [1, Sec. 4].

Lemma 3.1 *Let $\{u_k\}_{k\in\mathbf{N}_0}, \{v_k\}_{k\in\mathbf{N}_0}$ be non-negative sequences (if $v_k = 0$ we set $v_k^{-1} = 0$). Then*

a)
$$\sum_{k=0}^{\infty}\Big|\sum_{j=0}^{k} a_j\Big|^2 u_k \leq C \sup_N \Big(\sum_{k=N}^{\infty} u_k \sum_{k=0}^{N} v_k^{-1}\Big) \sum_{j=0}^{\infty} |a_j|^2 v_j.$$

b)
$$\sum_{k=0}^{\infty}\Big|\sum_{j=k}^{\infty} a_j\Big|^2 u_k \leq C \sup_N \Big(\sum_{k=0}^{N} u_k \sum_{k=N}^{\infty} v_k^{-1}\Big) \sum_{j=0}^{\infty} |a_j|^2 v_j.$$

Proof of Theorem 1.4. Using (9) and the operator T_m defined in (7), we obtain

$$\int_0^\infty |T_m f(x) e^{-x/2}|^2 x^{\alpha+1} dx \approx \sum_{k=0}^{\infty} A_k^{\alpha+1} |\Delta(m_k \hat{f}_\alpha(k))|^2.$$

Since

$$\Delta(m_k \hat{f}_\alpha(k)) = m_k \Delta \hat{f}_\alpha(k) + \hat{f}_\alpha(k+1)\Delta m_k \tag{11}$$

we first observe that

$$\sum_{k=0}^{\infty} A_k^{\alpha+1} |m_k|^2 |\Delta \hat{f}_\alpha(k)|^2 \leq \|m\|_\infty^2 \sum_{k=0}^{\infty} A_k^{\alpha+1} |\Delta \hat{f}_\alpha(k)|^2 \leq C \|m\|_\infty^2 \|f\|^2_{L^2_{w(\alpha+1)}}.$$

To dominate the term containing Δm_k we deduce from (8) that for $\alpha \geq 0$ the Fourier Laguerre coefficients tend to zero as $k \to \infty$. Hence

$$\sum_{k=0}^{\infty} A_k^{\alpha+1} |\hat{f}_\alpha(k+1)\Delta m_k|^2 = \sum_{k=0}^{\infty} A_k^{\alpha+1} |\Delta m_k|^2 \Big|\sum_{j=k+1}^{\infty} \Delta \hat{f}_\alpha(j)\Big|^2 =: I.$$

In order to apply Lemma 3.1 b), we choose $u_k = A_k^{\alpha+1}|\Delta m_k|^2$ and $v_k = A_k^{\alpha+1}$, and observe that when $M \in \mathbf{N}$, $2^{M-1} \leq N < 2^M$, we have that

$$\begin{aligned}\Big(\sum_{k=0}^{N} u_k \sum_{k=N}^{\infty} v_k^{-1}\Big) &\leq C(N+1)^{-\alpha} \sum_{j=0}^{M} \sum_{k=2^j-1}^{2^{j+1}-2} (k+1)|\Delta m_k|^2 \frac{A_k^{\alpha+1}}{k+1} \\ &\leq C(N+1)^{-\alpha} \sum_{j=0}^{M} (2^j)^\alpha \|m\|_{2,1}^2 \leq C\|m\|_{2,1}^2\end{aligned}$$

uniformly in N if $\alpha > 0$. Then Lemma 3.1 b) gives

$$I \leq C\|m\|_{2,1}^2 \sum_{j=0}^{\infty} A_j^{\alpha+1} |\Delta \hat{f}_\alpha(j)|^2 \leq C\|m\|_{2,1}^2 \|f\|_{L^2_{w(\alpha+1)}}^2$$

by (9). Thus there remains to consider the case $-1 < \alpha < 0$. For the same choice of u_k and v_k one easily obtains

$$\Big(\sum_{k=N}^{\infty} u_k \sum_{k=0}^{N} v_k^{-1} \Big) \leq C\|m\|_{2,1}^2 .$$

Now assume that $\hat{f}(0) = 0$. Then we have

$$\sum_{k=0}^{\infty} A_k^{\alpha+1} |\hat{f}_\alpha(k+1)\Delta m_k|^2 = \sum_{k=0}^{\infty} A_k^{\alpha+1} |\Delta m_k|^2 \Big| \sum_{j=0}^{k} \Delta \hat{f}_\alpha(j) \Big|^2 \leq C\|m\|_{2,1}^2 \|f\|_{L^2_{w(\alpha+1)}}^2 ,$$

where the last estimate follows by Lemma 3.1 a); thus Theorem 1.4 is established.

The **proof of Theorem 1.5** is essentially contained in [6]. As in [6], consider a monotone decreasing C^∞-function $\phi(x)$ with

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x \geq 4 \end{cases} , \quad \phi_i(x) = \phi(x/2^i).$$

Then the $\phi_i(k)$ are the Fourier Laguerre coefficients of an $L^2_{w(\alpha+1)}$-function $\Phi^{(i)}$ with norm $\|\Phi^{(i)}\|_{L^2_{w(\alpha+1)}} \leq C\,(2^i)^{\alpha/2}$ and

$$\sum_{k=2^i}^{2^{i+1}} A_k^{\alpha+1} |\Delta m_k|^2 = \sum_{k=2^i}^{2^{i+1}} A_k^{\alpha+1} |\Delta(m_k \phi_i(k))|^2 \leq \sum_{k=0}^{2^{i+2}} A_k^{\alpha+1} |\Delta(m_k \phi_i(k))|^2$$

$$\leq C\|T_m \Phi^{(i)}\|_{L^2_{w(\alpha+1)}}^2 \leq C\|m\|_{M^2_{\alpha;\,\alpha+1}}^2 \|\Phi^{(i)}\|_{L^2_{w(\alpha+1)}}^2 \leq C 2^{i\alpha} \|m\|_{M^2_{\alpha;\,\alpha+1}}^2 .$$

This immediately leads to

$$\|m\|_\infty + \left(\sum_{2^i}^{2^{i+1}} |(k+1)\Delta m_k|^2 \frac{1}{k+1} \right)^{1/2} \leq C\|m\|_{M^2_{\alpha;\,\alpha+1}},$$

uniformly in i, since by [6, (10)] there holds $\|m\|_\infty \leq C\|m\|_{M^2_{\alpha;\,\alpha+1}}$; thus Theorem 1.5 is established.

Remark. 3) (Added on Aug. 10, 1994) The characterization (6) can easily be extended to

$$M^2_{\alpha,\alpha+l} = wbv_{2,l}, \quad \alpha > -1, \quad \alpha \neq 0, \ldots, l-1, \; l \in \mathbf{N}. \tag{12}$$

In the case $\alpha < l-1$ the multiplier operator is defined only on the subspace $\{f \in L^2_{w(\alpha+l)} : \hat{f}_\alpha(k) = 0,\ 0 \le k < (l-1-\alpha)/2\}$.

The necessity part carries over immediately (see also [6]). The sufficiency part will be proved by induction. Thus suppose that (12) is true for $l = 1, \ldots, n$ and α's as indicated. Then, as in the case $n = 1$, by (9)

$$\int_0^\infty |T_m f(x) e^{-x/2}|^2 x^{\alpha+n+1} dx \approx \sum_{k=0}^\infty A_k^{\alpha+n+1} |\Delta^n \Delta(m_k \hat{f}_\alpha(k))|^2$$

$$\le C \sum_{k=0}^\infty A_k^{\alpha+n+1} |\Delta^n (m_k \Delta \hat{f}_\alpha(k))|^2 + C \sum_{k=0}^\infty A_k^{\alpha+n+1} |\Delta^n (\hat{f}_\alpha(k+1) \Delta m_k)|^2 =: I + II$$

By the assumption and (10)

$$I \le C \|m\|^2_{wbv_{2,n}} \sum_{k=0}^\infty A_k^{\alpha+n+1} |\Delta^n \hat{f}_{\alpha+1}(k))|^2 \le C \|m\|^2_{wbv_{2,n+1}} \int_0^\infty |f(x) e^{-x/2}|^2 x^{\alpha+n+1} dx$$

on account of the embedding properties of the wbv–spaces [5]. Analogously II can be estimated by

$$II \le C \|\{(k+1)\Delta m_k\}\|^2_{wbv_{2,n}} \sum_{k=0}^\infty A_k^{\alpha+n+1} \Big|\Delta^n \Big(\frac{\hat{f}_\alpha(k+1)}{k+1}\Big)\Big|^2 .$$

By the Leibniz formula for differences there holds

$$\begin{aligned}
\Delta^n \Big(\frac{\hat{f}_\alpha(k+1)}{k+1}\Big) &\le C \sum_{j=0}^n |\Delta^j \hat{f}_\alpha(k+1)|\, |\Delta^{n-j} \frac{1}{j+k+1}| \\
&\le C \sum_{j=0}^n (j+k+1)^{j-n-1} |\Delta^j \hat{f}_\alpha(k+1)| .
\end{aligned}$$

Hence we have to dominate for $j = 0, \ldots, n$

$$II_j := \sum_{k=0}^\infty A_k^{\alpha-n-1+2j} |\Delta^j \hat{f}_\alpha(k+1)|^2 .$$

If $\alpha > n$ then $c_j := -\alpha - 2j + n + 1 < 1$ for all $j = 0, \ldots, n$, $\Delta^j \hat{f}_\alpha(k+1) = \sum_{i=k+1}^\infty \Delta^{j+1} \hat{f}_\alpha(i)$, and we can apply [8, Theorem 346] repeatedly to obtain

$$II_j \le C \sum_{k=0}^\infty A_k^{\alpha-n-1+2j} |(k+1)\Delta^{j+1} \hat{f}_\alpha(k+1)|^2 \approx \sum_{k=0}^\infty A_k^{\alpha-n+2j+1} |\Delta^{j+1} \hat{f}_\alpha(k+1)|^2$$

$$\le \ldots \le C \sum_{k=0}^\infty A_k^{\alpha+n+1} |\Delta^{n+1} \hat{f}_\alpha(k+1)|^2 \le C \int_0^\infty |f(x) e^{-x/2}|^2 x^{\alpha+n+1} dx .$$

Since $\|\{(k+1)\Delta m_k\}\|_{wbv_{2,n}} \leq C\|m\|_{wbv_{2,n+1}}$, this gives the assertion for the weight x^{n+1} in the case $\alpha > n$.

If $\alpha < n$, $\alpha \neq 0, \ldots, n$, then some $c_j > 1$. For the application of [8, Theorem 346] one needs $c_j \neq 1$; this is guaranteed by the hypothesis $\alpha \neq 0, \ldots, n$ (in the case of an additional weight x^{n+1}). For the j for which $c_j > 1$ we have to use the representation

$$\Delta^j \hat{f}_\alpha(k+1) = -\sum_{i=0}^{k} \Delta^{j+1} \hat{f}_\alpha(i), \quad \text{if } \Delta^j \hat{f}_\alpha(0) = 0,$$

i.e., the first $(j+1)$ Fourier-Laguerre coefficients have to vanish to ensure this representation. But $0 \leq j \leq j_0$, where j_0 is choosen in such a way that $c_{j_0} > 1$ and $c_{j_0+1} < 1$, hence $j_0 = [(n-\alpha)/2]$ (with respect to the additional weight x^{n+1}); here we used the standard notation for $[a]$, $a \in \mathbf{R}$, to be the greatest integer $\leq a$. Hence the condition that the first $[(n-\alpha)/2]+1$ Fourier-Laguerre coefficients have to vanish is needed if the additional weight is x^{n+1}. A repeated application of [8, Theorem 346] with appropriate $c > 1$ or $c < 1$ now gives the assertion.

References

[1] K. F. Andersen and H.P. Heinig, Weighted norm inequalities for certain integral operators, *SIAM J. Math. Anal.,* 14 (1983), 834 – 844.

[2] R. Askey and J. Fitch, Integral representations for Jacobi polynomials and some applications, *J. Math. Anal. Appl.,* 26 (1969), 411 – 437.

[3] K. de Leeuw, On L^p multipliers, *Ann. of Math.,* 81 (1965), 364 – 379.

[4] G. Gasper, K. Stempak, and W. Trebels, Fractional integration for Laguerre expansions (to appear).

[5] G. Gasper and W. Trebels, A characterization of localized Bessel potential spaces and applications to Jacobi and Hankel multipliers, *Studia Math.,* 65 (1979), 243 – 278.

[6] G. Gasper and W. Trebels, On necessary multiplier conditions for Laguerre expansions, *Canad. J. Math.,* 43 (1991), 1228 – 1242.

[7] E. Görlich and C. Markett, A convolution structure for Laguerre series, *Indag. Math.,* 44 (1982), 161 – 171.

[8] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, 1964.

[9] Y. Kanjin, A transplantation theorem for Laguerre series, *Tohoku Math. J.,* 43 (1991), 537 – 555.

[10] C. Markett, Mean Cesàro summability of Laguerre expansions and norm estimates with shifted parameter, *Anal. Math.,* 8 (1982), 19 – 37.

[11] B. Muckenhoupt, Hardy's inequality with weights, *Studia Math.,* 44 (1972), 31 – 38.

[12] B. Muckenhoupt, Transplantation theorems and multiplier theorems for Jacobi series, *Mem. Amer. Math. Soc.,* 64 (1986), no. 356.

[13] B. Muckenhoupt, R.L Wheeden, and Wo-Sang Young, L^2 multipliers with power weights, *Adv. in Math.,* 49 (1983), 170 –216.

[14] K. Stempak and W. Trebels, On weighted transplantation and multipliers for Laguerre expansions, *Math. Ann.* (to appear).

[15] G. Szegö, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. 23, Providence, R.I., 1975.

[16] S. Thangavelu, Transplantation, summability and multipliers for multiple Laguerre expansions, *Tohoku Math. J.,* 44 (1992), 279 – 298.

[17] M. Zafran, Interpolation of multiplier spaces, *Amer. J. Math.,* 105 (1983), 1405 – 1416.