q-Extensions of Barnes', Cauchy's, and Euler's beta integrals*

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Abstract. It is shown how Cauchy's residue theorem and certain summation formulas for basic hypergeometric series can be used to further extend some previous q-extensions of Barnes', Cauchy's, and Euler's beta integrals. In addition, related basic contour integrals are evaluated, a general transformation formula for basic hypergeometric series is derived and some directions for future research are pointed out.

1. Introduction. The beta function is usually defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{1.1}$$

where $\mathrm{Re}(x) > 0$, $\mathrm{Re}(y) > 0$, and $\Gamma(x)$ is the gamma function. The above integral is called Euler's beta integral on the the interval $[0,1]$. Euler's beta integral on the interval $[0,\infty)$ is the integral

$$\int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}}\,ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \mathrm{Re}(x) > 0, \quad \mathrm{Re}(y) > 0. \tag{1.2}$$

This formula follows from (1.1) by setting $t = s/(1+s)$. The integrals in (1.1) and (1.2) can be converted to beta integrals on any finite or half infinite interval, respectively, by a linear change in the variable of integration.

In 1825 Cauchy [12,13] showed that if $\mathrm{Re}(c) > 0$, $\mathrm{Re}(d) > 0$ and $\mathrm{Re}(x+y) > 1$, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dt}{(1+ict)^x(1-idt)^y} = \frac{\Gamma(x+y-1)(1+d/c)^{1-y}(1+c/d)^{1-x}}{\Gamma(x)\Gamma(y)(c+d)} \tag{1.3}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dt}{(1+ict)^x(1+idt)^y} = 0. \tag{1.4}$$

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Askey [5] called these integrals "beta integrals" because, as in Euler's beta integrals, they are integrals of the product of two powers of linear functions on an appropriate contour.

Barnes [11] showed in 1908 that if $\mathrm{Re}(a, b, c, d) > 0$, then

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(a+it)\Gamma(b+it)\Gamma(c-it)\Gamma(d-it)dt \\ &= \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}, \end{aligned} \tag{1.5}$$

which is usually referred to as Barnes' lemma. Askey and Roy [8] called the integral in (1.5) Barnes' beta integral because, as they showed, (1.1) is a limit case of (1.5).

In [33] Wilson gave his solution to a problem proposed by Askey of proving the following q-extension of (1.3): if $\mathrm{Re}(c) > 0$, $\mathrm{Re}(d) > 0$ and $\mathrm{Re}(x + y) > 1$, then

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(-ictq^x, idtq^y; q)_\infty}{(-ict, idt; q)_\infty}\,dt \\ &= \frac{\Gamma_q(x+y-1)(-dq^y/c, -cq^x/d; q)_\infty}{\Gamma_q(x)\Gamma_q(y)(-dq/c, -cq/d; q)_\infty (c+d)}, \end{aligned} \tag{1.6}$$

where $0 < q < 1$,

$$\begin{aligned} (a; q)_\infty &= \prod_{k=0}^{\infty}(1 - aq^k), \\ (a_1, a_2, \ldots, a_m; q)_\infty &= (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty, \end{aligned}$$

and the q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty}(1-q)^{1-x}.$$

Another q-extension of Cauchy's beta integral is given in [5].

Askey and Roy [8] derived the formula

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(fe^{i\theta}/d, dqe^{-i\theta}/f, cfe^{-i\theta}, qe^{i\theta}/cf; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{-i\theta}, de^{-i\theta}; q)_\infty}\,d\theta \\ &= \frac{(abcd, f, q/f, cf/d, dq/cf; q)_\infty}{(q, ac, ad, bc, bd; q)_\infty}, \end{aligned} \tag{1.7}$$

where $\max(|a|, |b|, |c|, |d|, |q|) < 1$ and $cdf \neq 0$, and showed that (1.5) is a limit case of (1.7). They also derived another q-extension of (1.5) involving an integral from minus infinity to infinity which is different from a q-extension derived over seventy years earlier by Watson [31].

Formulas (1.6) and (1.7) were derived by the standard method of applying Cauchy's residue theorem (see, e.g., [32, p. 112] and [27, Chapter 4]) and summing the residues that arose. In this paper we shall show that Cauchy's residue theorem can also be used to extend (1.7) to

$$\begin{aligned}(1.8)\qquad &\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\gamma e^{-i\theta},\alpha\beta q e^{-i\theta}/\gamma,\delta e^{i\theta},qe^{i\theta}/\gamma,\gamma e^{i\theta}/\alpha\beta;q)_\infty}{(\alpha e^{-i\theta},\beta e^{-i\theta},ae^{i\theta},be^{i\theta},ce^{i\theta};q)_\infty}\,d\theta\\ &=\frac{(\gamma/\alpha,\alpha q/\gamma,\gamma/\beta,\beta q/\gamma,\delta/a,\delta/b,\delta/c;q)_\infty}{(q,a\alpha,a\beta,b\alpha,b\beta,c\alpha,c\beta;q)_\infty}\end{aligned}$$

where $\delta = abc\alpha\beta$, $\max(|a|,|b|,|c|,|\alpha|,|\beta|,|q|) < 1$ and $abc\alpha\beta\gamma \neq 0$, to derive extensions of (1.6) and (1.8), to evaluate some related integrals, and to derive a general transformation for basic hypergeometric series. Formula (1.7) is the $c \to 0$ limit case of (1.8). See [3, 6, 8, 17] for some q-extensions of (1.2) and see [4, 21, 25, 29] for some multidimensional extensions of (1.1). Unless stated otherwise, it is assumed below that q is a complex number and $0 < |q| < 1$.

2. Summation formulas. In order to sum the residues that arise, we shall need to use some summation formulas for basic hypergeometric series.

Letting

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$$

denote the q-shifted factorial and setting

$$(a_1,a_2,\ldots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n,$$

a ${}_{r+1}\phi_r$ basic hypergeometric series is defined by

$$(2.1)\qquad {}_{r+1}\phi_r\left[\begin{matrix}a_1,\ldots,a_{r+1}\\ b_1,\ldots,b_r\end{matrix};q,z\right] = \sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_{r+1};q)_k}{(q,b_1,\ldots,b_r;q)_k}z^k,\qquad |z|<1.$$

In (2.1) it is assumed that no denominator parameter is 1 or a negative integer power of q. Note that since $(q^{-n};q)_k = 0$ for $k = n+1, n+2, \ldots$, this series terminates if one of its numerator parameters $a_1,\ldots,a_{r+1}$ is 1 or a negative integer power of q.

In 1843 Cauchy [14] derived the terminating case of the formula

$$(2.2)\qquad {}_1\phi_0\left[\begin{matrix}a\\ -\end{matrix};q,z\right] = \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k = \frac{(az;q)_\infty}{(z;q)_\infty},\qquad |z|<1,$$

which sums a ${}_1\phi_0$ series and is a q-analogue of the binomial theorem (also see Bailey [9], Slater [30], and Gasper and Rahman [19]). By replacing a in (2.2) by q^a, letting $q \to 1$ and using the binomial theorem it can be shown that

$$(2.3)\qquad \lim_{q\to 1^-}\frac{(q^a z;q)_\infty}{(q;q)_\infty} = (1-z)^{-a},\qquad |z|<1, -\infty<a<\infty.$$

For a rigorous proof of (2.3) and of

$$\lim_{q\to 1^-} \Gamma_q(x) = \Gamma(x), \qquad 0 < x < \infty, \tag{2.4}$$

see Koornwinder [26]. Also see [1, 2, 3, 7].

Heine [22] showed in 1847 that

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \qquad |c/ab| < 1, \tag{2.5}$$

which is a q-analogue of Gauss' summation formula for a ${}_2F_1(1)$ series and contains (2.2) as a limit case. Wilson's proof of (1.6) used (2.5) to sum the residues at the poles of the integrand that were in the upper half plane. Heine's sum is the $n \to \infty$ limit case of

$$ {}_3\phi_2\left[\begin{matrix} a, b, q^{-n} \\ c, abq^{1-n}/c \end{matrix}; q, q\right] = \frac{(c/a, c/b; q)_n}{(c, c/ab; q)_n}, \qquad n = 0, 1, \ldots, \tag{2.6}$$

which was derived in 1910 by Jackson [23] and is sometimes called the q-Saalschütz formula. In 1921 Jackson [24] extended (2.6) to

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix}; q, q\right] \\ &= \frac{(aq, aq/bc, aq/cd, aq/bd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}, \qquad n = 0, 1, \ldots, \end{aligned} \tag{2.7}$$

where $a^2 q = bcdeq^{-n}$.

In their proof of (1.7) Askey and Roy derived and used a summation formula which is a limit case of the following nonterminating extension of (2.6):

$$\begin{aligned} &{}_3\phi_2\left[\begin{matrix} a, b, c \\ e, f \end{matrix}; q, q\right] + \frac{(a, b, c, q/e, fq/e; q)_\infty}{(e/q, aq/e, bq/e, cq/e, f; q)_\infty} \\ &\cdot {}_3\phi_2\left[\begin{matrix} aq/e, bq/e, cq/e \\ fq/e, q^2/e \end{matrix}; q, q\right] = \frac{(f/a, f/b, f/c, q/e; q)_\infty}{(aq/e, bq/e, cq/e, f; q)_\infty}, \end{aligned} \tag{2.8}$$

where $ef = abcq$. This formula is a limit case of Bailey's [10] nonterminating extension of (2.7):

$$\begin{aligned} &{}_8\phi_7\left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e, f \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix}; q, q\right] \\ &+ \frac{(bq/c, bq/d, bq/e, bq/f, aq, c, d, e, f, b/a; q)_\infty}{(b^2q/a, bc/a, bd/a, be/a, bf/a, aq/c, aq/d, aq/e, aq/f, a/b; q)_\infty} \\ &\cdot {}_8\phi_7\left[\begin{matrix} b^2/a, qba^{-\frac{1}{2}}, -qba^{-\frac{1}{2}}, b, bc/a, bd/a, be/a, bf/a \\ ba^{-\frac{1}{2}}, -ba^{-\frac{1}{2}}, bq/a, bq/c, bq/d, bq/e, bq/f \end{matrix}; q, q\right] \\ &= \frac{(aq, b/a, aq/de, aq/ce, aq/cd, aq/cf, aq/df, aq/ef; q)_\infty}{(aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a, bf/a; q)_\infty}, \end{aligned} \tag{2.9}$$

where $a^2q = bcdef$.

We shall also use the author's [15] summation formula

$$
(2.10)\qquad {}_{r+2}\phi_{r+1}\left[\begin{matrix} a, b, b_1q^{m_1}, \ldots, b_rq^{m_r} \\ bq, b_1, \ldots, b_r \end{matrix}; q, a^{-1}q^{1-(m_1+\cdots+m_r)}\right]
$$
$$
= \frac{(q, bq/a; q)_\infty (b_1/b; q)_{m_1} \cdots (b_r/b; q)_{m_r}}{(bq, q/a; q)_\infty (b_1; q)_{m_1} \cdots (b_r; q)_{m_r}} b^{m_1+\cdots+m_r},
$$

where $m_1, \ldots, m_r$ are nonnegative integers and $|a^{-1}q^{1-(m_1+\cdots+m_r)}| < 1$. The $b \to 0$ limit case of (2.10) was recently rediscovered by Gustafson in his work [20] on multivariable orthogonal polynomials. Some applications of (2.10) to orthogonal functions are given in [16].

3. Derivation of (1.8) and an extension.

Suppose that the function

$$
(3.1)\qquad f(z) = \frac{(\gamma/z, \delta/z, a_1z, \ldots, a_rz; q)_\infty}{(\alpha/z, \beta/z, b_1z, \ldots, b_rz; q)_\infty}
$$

has only simple poles and consider the contour integral

$$
(3.2)\qquad I = \frac{1}{2\pi i}\int_C f(z)\frac{dz}{z},
$$

where the contour C is a deformation of the (positively oriented) unit circle so that the poles of $1/(b_1z, \ldots, b_rz; q)_\infty$ lie outside the contour and the origin and poles of $1/(\alpha/z, \beta/z; q)_\infty$ lie inside the contour. Let ρ be a positive number such that $|\rho q^j|$ does not equal $|\alpha|, |\beta|, |b_1^{-1}|, \ldots, |b_r^{-1}|$ for $j = 0, \pm1, \pm2, \ldots$, and let C_k be the circle $|z| = |\rho q^k|$, where k is a positive integer which is so large that C_k lies inside C. Then C_k does not pass through any of the poles of $f(z)$. Also let M_k equal the maximum value of $|f(z)|$ on C_k. From

$$
f(qz) = \frac{(1-\gamma/qz)(1-\delta/qz)(1-b_1z)\cdots(1-b_rz)}{(1-\alpha/qz)(1-\beta/qz)(1-a_1z)\cdots(1-a_rz)} f(z)
$$

and the fact that

$$
(3.3)\qquad \left|\frac{1-\gamma/z}{1-\alpha/z}\right| = \left|\frac{\gamma}{\alpha}\right|(1+O(|z|)) \qquad \text{as } z \to 0,
$$

we obtain that

$$
(3.4)\qquad M_{k+1} = \left|\frac{\gamma\delta}{\alpha\beta}\right|(1+O(|q|^k))M_k \qquad \text{as } k \to \infty,
$$

and hence $M_k \to 0$ as $k \to \infty$ if $|\gamma\delta/\alpha\beta| < 1$. Therefore, if $|\gamma\delta/\alpha\beta| < 1$, then

$$
(3.5)\qquad \lim_{k\to\infty} \frac{1}{2\pi i}\int_{C_k} f(z)\frac{dz}{z} = \lim_{k\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\rho|q|^k e^{i\theta})d\theta = 0,
$$

and so, by applying Cauchy's residue theorem to the region between C and C_k and letting $k \to \infty$, we obtain that the integral I equals the sum of the residues of $f(z)/z$ at the poles of $1/(\alpha/z, \beta/z; q)_\infty$.

Since the residue of $1/(\alpha/z; q)_\infty$ at the pole $z = \alpha q^n$ equals

$$\frac{(-1)^n \alpha q^{2n+\binom{n}{2}}}{(q;q)_n (q;q)_\infty},$$

where $\binom{n}{2} = n(n-1)/2$ and $n = 0, 1, \ldots$, it follows by using the identities

$$(aq^n; q)_\infty = (a;q)_\infty/(a;q)_n, \tag{3.6}$$

$$(aq^{-n}; q)_n = (-a/q)^n q^{-\binom{n}{2}} (q/a; q)_n, \tag{3.7}$$

that

$$\begin{aligned} &\frac{1}{2\pi i}\int_C \frac{(\gamma/z, \delta/z, a_1 z, \ldots, a_r z; q)_\infty}{(\alpha/z, \beta/z, b_1 z, \ldots, b_r z; q)_\infty}\frac{dz}{z} \\ &= \frac{(\gamma/\alpha, \delta/\alpha, a_1\alpha, \ldots, a_r\alpha; q)_\infty}{(q, \beta/\alpha, b_1\alpha, \ldots, b_r\alpha; q)_\infty} \\ &\cdot {}_{r+2}\phi_{r+1}\left[\begin{matrix}\alpha q/\gamma, \alpha q/\delta, b_1\alpha, \ldots, b_r\alpha \\ \alpha q/\beta, a_1\alpha, \ldots, a_r\alpha\end{matrix}; q, \frac{\gamma\delta}{\alpha\beta}\right] \\ &+ \frac{(\gamma/\beta, \delta/\beta, a_1\beta, \ldots, a_r\beta; q)_\infty}{(q, \alpha/\beta, b_1\beta, \ldots, b_r\beta; q)_\infty} \\ &\cdot {}_{r+2}\phi_{r+1}\left[\begin{matrix}\beta q/\gamma, \beta q/\delta, b_1\beta, \ldots, b_r\beta \\ \beta q/\alpha, a_1\beta, \ldots, a_r\beta\end{matrix}; q, \frac{\gamma\delta}{\alpha\beta}\right], \end{aligned} \tag{3.8}$$

when $|\gamma\delta/\alpha\beta| < 1$. The cases when the integrand in (3.8) has multiple poles can be treated as limit cases of (3.8). From (2.8) and the $r = 3$ case of (3.8) we get

$$\begin{aligned} &\frac{1}{2\pi i}\int_C \frac{(\gamma/z, \alpha\beta q/\gamma z, \delta z, qz/\gamma, \gamma z/\alpha\beta; q)_\infty}{(\alpha/z, \beta/z, az, bz, cz; q)_\infty}\frac{dz}{z} \\ &= \frac{(\gamma/\alpha, \alpha q/\gamma, \gamma/\beta, \beta q/\gamma, \delta/a, \delta/b, \delta/c; q)_\infty}{(q, a\alpha, a\beta, b\alpha, b\beta, c\alpha, c\beta; q)_\infty}, \end{aligned} \tag{3.9}$$

where $\delta = abc\alpha\beta$, $abc\alpha\beta\gamma \neq 0$,

$$a\alpha, a\beta, b\alpha, b\beta, c\alpha, c\beta \neq q^{-n}, \qquad n = 0, 1, \ldots$$

and the contour is as described below (3.2). When $\max(|a|, |b|, |c|, |\alpha|, |\beta|) < 1$, C can be taken to be the unit circle and then (3.9) reduces to (1.8) by setting $z = e^{i\theta}$.

Similarly, using Bailey's summation formula (2.9) and (3.8), we find that (3.9) can be extended to

$$\begin{aligned} &\frac{1}{2\pi i}\int_C \frac{(\gamma/z, \alpha\beta q/\gamma z, za^{\frac12}, -za^{\frac12}, qz/\gamma, \gamma z/\alpha\beta; q)_\infty}{(\alpha/z, \beta/z, qza^{\frac12}, -qza^{\frac12}, a\alpha z, a\beta z; q)_\infty} \\ &\cdot \frac{(aqz/b, aqz/c, aqz/d, aqz/f; q)_\infty}{(bz, cz, dz, fz; q)_\infty}\frac{dz}{z} \\ &= \frac{(\gamma/\alpha, \alpha q/\gamma, \gamma/\beta, \beta q/\gamma, aq/cd, aq/bd, aq/bc, aq/bf, aq/cf, aq/df; q)_\infty}{(q, a\alpha\beta, b\alpha, b\beta, c\alpha, c\beta, d\alpha, d\beta, f\alpha, f\beta; q)_\infty}, \end{aligned} \tag{3.10}$$

where $aq = bcdf\alpha\beta$, $bcdf\alpha\beta\gamma \neq 0$.

$$a\alpha\beta, b\alpha, b\beta, c\alpha, c\beta, d\alpha, d\beta, f\alpha, f\beta \neq q^{-n}, \qquad n = 0, 1, \ldots$$

and the contour C is a deformation of the positively oriented unit circle so that the poles of $1/(bz, cz, dz, fz, a\alpha z, a\beta z; q)_\infty$ lie outside the contour and the origin and poles of $1/(\alpha/z, \beta/z; q)_\infty$ lie inside the contour. The integral in (3.10) can be slightly simplified by using the fact that

$$\frac{(za^{\frac{1}{2}}, -za^{\frac{1}{2}}; q)_\infty}{(qza^{\frac{1}{2}}, -qza^{\frac{1}{2}}; q)_\infty} = 1 - az^2.$$

When $\beta = \delta$, (3.8) reduces to

$$\begin{aligned} \frac{1}{2\pi i} \int_C \frac{(\gamma/z, a_1 z, \ldots, a_r z; q)_\infty}{(\alpha/z, b_1 z, \ldots, b_r z; q)_\infty} \frac{dz}{z} &\\ = \frac{(\gamma/\alpha, a_1\alpha, \ldots, a_r\alpha; q)_\infty}{(q, b_1\alpha, \ldots, b_r\alpha; q)_\infty} &\\ \cdot {}_{r+1}\phi_r \left[\begin{matrix} \alpha q/\gamma, b_1\alpha, \ldots, b_r\alpha \\ a_1\alpha, \ldots, a_r\alpha \end{matrix} ; q, \frac{\gamma}{\alpha} \right], & \end{aligned} \tag{3.11}$$

provided $|\gamma/\alpha| < 1$, the poles of $1/(b_1 z, \ldots, b_r z; q)_\infty$ lie outside of the contour and the origin and poles of $1/(\alpha/z; q)_\infty$ lie inside the contour.

Using (2.5), the $r = 1$ case of (3.11) gives

$$\frac{1}{2\pi i} \int_C \frac{(\gamma/z, bqz; q)_\infty}{(\alpha/z, bz; q)_\infty} \frac{dz}{z} = \frac{(b\gamma; q)_\infty}{(b\alpha; q)_\infty}, \qquad \left|\frac{\gamma}{\alpha}\right| < 1. \tag{3.12}$$

Unfortunately, in order to be able to apply (2.6) or (2.7) to (3.11) we need to have $\gamma = \alpha q$; but then the ${}_{r+1}\phi_r$ series equals 1 and so (3.11) reduces to

$$\frac{1}{2\pi i} \int_C \frac{(a_1 z, \ldots, a_r z; q)_\infty}{(b_1 z, \ldots, b_r z; q)_\infty} \frac{dz}{z - \alpha} = \frac{(a_1\alpha, \ldots, a_r\alpha; q)_\infty}{(b_1\alpha, \ldots, b_r\alpha; q)_\infty}. \tag{3.13}$$

which is a special case of Cauchy's integral formula. However, if we observe that by setting $e = a^2 q^{n+1}/bcd$ in (2.7) and letting $n \to \infty$, we get the summation formula

$$\begin{aligned} {}_6\phi_5 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d \end{matrix} ; q, \frac{aq}{bcd} \right] &\\ = \frac{(aq, aq/bc, aq/cd, aq/bd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty} \qquad \left|\frac{aq}{bcd}\right| < 1, & \end{aligned} \tag{3.14}$$

we can apply this sum to (3.11) to get

$$\begin{aligned} \frac{1}{2\pi i} \int_C \frac{(\gamma/z, q^2 z/a\alpha\gamma, q^2 z/b\alpha\gamma, q^2/c\alpha\gamma; q)_\infty}{(\alpha/z, az, bz, cz; q)_\infty} \frac{(1 - qz^2/\alpha\gamma)dz}{z} &\\ = \frac{(\alpha q/\gamma, a\gamma, b\gamma, c\gamma; q)_\infty}{(q, a\alpha, b\alpha, c\alpha; q)_\infty}, & \end{aligned} \tag{3.15}$$

where $q^2 = abc\alpha\gamma^2$, $|\gamma/\alpha| < 1$, the poles of $1/(az, bz, cz; q)_\infty$ lie outside the contour and the origin and poles of $1/(\alpha/z; q)_\infty$ lie inside the contour.

In addition, since (3.11) gives

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_C \frac{(\gamma/z, bqz, qz/\gamma q^m, a_1 z, \ldots, a_r z; q)_\infty}{(\alpha/z, az, bz, a_1 z q^{m_1}, \ldots, a_r z q^{m_r}; q)_\infty}\frac{dz}{z} \\
&= \frac{(\gamma/\alpha, b\alpha q, \alpha q/\gamma q^m, a_1\alpha, \ldots, a_r\alpha; q)_\infty}{(q, a\alpha, b\alpha, a_1\alpha q^{m_1}, \ldots, a_r\alpha q^{m_r}; q)_\infty} \\
&\cdot {}_{r+3}\phi_{r+2}\left[\begin{matrix} a\alpha, b\alpha, \alpha q/\gamma, a_1\alpha q^{m_1}, \ldots, a_r\alpha q^{m_r} \\ b\alpha q, \alpha q/\gamma q^m, a_1\alpha, \ldots, a_r\alpha \end{matrix}; q, \frac{\gamma}{\alpha}\right],
\end{aligned}
\tag{3.16}
$$

we can apply (2.10) to the above series and use (3.6) to obtain

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_C \frac{(\gamma/z, bqz, qz/\gamma; q)_\infty}{(\alpha/z, az, bz; q)_\infty}(qz/\gamma q^m; q)_m (a_1 z; q)_{m_1}\cdots(a_r z; q)_{m_r}\frac{dz}{z} \\
&= \frac{(\gamma/\alpha, \alpha q/\gamma, bq/a; q)_\infty}{(a\alpha, q/a\alpha, b\alpha; q)_\infty}(q/b\gamma q^m; q)_m (a_1/b; q)_{m_1}\cdots(a_r/b; q)_{m_r}(b\alpha)^{m+m_1+\cdots+m_r},
\end{aligned}
\tag{3.17}
$$

provided $|\gamma/\alpha| < 1$, where $m, m_1, \ldots, m_r$ are nonnegative integers, $q = a\gamma q^{m+m_1+\cdots+m_r}$ and C is a deformation of the unit circle so that the poles of $1/(az, bz; q)_\infty$ lie outside C and the origin and poles of $1/(\alpha/z; q)_\infty$ lie inside C.

Notice that if we let $q = e^{-t}$ with $t > 0$ and set $z = q^s$, then for the integral in (3.2) we obtain that

$$
\frac{1}{2\pi i}\int_C f(z)\frac{dz}{z} = \frac{t}{2\pi i}\int_{-i\pi/t}^{i\pi/t} f(q^s)ds,
\tag{3.18}
$$

where in the integral on the right side the contour of integration is along the imaginary axis with indentations, if necessary, to separate the increasing sequences of poles in $|\mathrm{Im}(s)| \le \pi/t$ from those that are decreasing. Basic contour integrals of the type on the right side of (3.18) were considered by Slater in Chapter 5 of [30] and used to derive several formulas involving sums of basic hypergeometric series.

4. Extensions of (1.6). Let $0 < q < 1$, $\mathrm{Re}(c, d_1, \ldots, d_r) > 0$,

$$
g(z) = \frac{(-iacz, ib_1 z, \ldots, ib_r z; q)_\infty}{(-icz, id_1 z, \ldots, id_r z; q)_\infty}
\tag{4.1}
$$

and

$$
J = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)dt.
\tag{4.2}
$$

This integral reduces to (1.6) when $r = 1$, $a = q^x$, $b_1 = dq^y$, and $d_1 = d$. The restriction that $0 < q < 1$ is used to ensure that the poles of $1/(-icz; q)_\infty$ lie in the upper half plane and the poles of $1/(id_1 z, \ldots, id_r z; q)_\infty$ lie in the lower half plane.

As in [33] let S_k, $k = 1, 2, \ldots$, be the semicircle in the upper half plane with center at the origin and radius $|c^{-1}q^{-k-1/2}|$, M_k be the maximum value of $g(z)$ on S_k, and let L_k be the length of S_k. From

$$g(z) = \frac{(1+iacz)(1-ib_1z)\cdots(1-ib_rz)}{(1+icz)(1-id_1z)\cdots(1-id_rz)}\, g(qz)$$

and

$$\left|\frac{1+bz}{1+dz}\right| = \left|\frac{b}{d}\right| (1 + O(|z|^{-1})) \qquad \text{as } |z| \to \infty,$$

it follows that

$$M_{k+1}L_{k+1} = \left|\frac{ab_1\cdots b_r}{qd_1\cdots d_r}\right| (1 + O(q^k))M_kL_k \qquad \text{as } k \to \infty,$$

and hence $M_kL_k \to 0$ as $k \to \infty$ if $|ab_1\cdots b_r/qd_1\cdots d_r| < 1$. Therefore, if $|ab_1\cdots b_r/qd_1\cdots d_r| < 1$, then

$$\lim_{k\to\infty} \int_{S_k} g(z)dz = 0,$$

and thus, by Cauchy's residue theorem, the integral J equals i times the sum of the residues of $g(z)$ at the poles of $1/(-icz;q)_\infty$. Since the residue of $1/(-icz;q)_\infty$ at the pole i/cq^n equals

$$\frac{(-1)^n q^{\binom{n}{2}}}{ic(q;q)_n(q;q)_\infty}$$

for $n = 0, 1, \ldots$, by using the identities (3.6) and (3.7) we obtain

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(-iact, ib_1t, \ldots, ib_rt;q)_\infty}{(-ict, id_1t, \ldots, id_rt;q)_\infty}\, dt \\ &= \frac{(a, -b_1/c, \ldots, -b_r/c;q)_\infty}{c(q, -d_1/c, \ldots, -d_r/c;q)_\infty} \\ &\cdot {}_{r+1}\phi_r\left[\begin{matrix} q/a, -cq/b_1, \ldots, -cq/b_r \\ -cq/d_1, \ldots, -cq/d_r \end{matrix}; q, \frac{ab_1\cdots b_r}{qd_1\cdots d_r}\right] \end{aligned} \tag{4.3}$$

if $|ab_1\cdots b_r/qd_1\cdots d_r| < 1$ and $\mathrm{Re}(c, d_1, \ldots, d_r) > 0$. The above integral equals zero if $\mathrm{Re}(-c, d_1, \ldots, d_r) > 0$, since then there are no poles in the upper half plane.

Application of the q-Saalschütz sum (2.6) to the $r = 2$ case of (4.3) gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(-ictq^{n+1}, ibdt, i\alpha t;q)_\infty}{(-ict, idt, ib\alpha tq^{n-1};q)_\infty}\, dt = \frac{(-\alpha/c, -bd/c;q)_\infty}{(c+d)(-dq/c, -b\alpha/cq;q)_\infty} \frac{(b, \alpha/d;q)_n}{(q, -cq/d;q)_n}, \tag{4.4}$$

where $\mathrm{Re}(c, d, b\alpha) > 0$ and $n = 0, 1, \ldots$. Formula (1.6) can be obtained as a limit case of (4.4) by letting $n \to \infty$ and then setting $b = q^y$ and $\alpha = -cq^x$. More generally, application of Jackson's sum (2.7) to (4.3) gives

$$\tag{4.5} \begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(-iact, iac^2t/\alpha, iac^2t/\beta, iac^2t/\gamma, iac^2t/\delta, iac^2t/\lambda; q)_\infty}{(-ict, i\alpha t, i\beta t, i\gamma t, i\delta t, i\lambda t; q)_\infty} \left(1+\frac{ac^2t^2}{q}\right) dt \\ &= \frac{(a/q, -ac/\alpha, -ac/\beta, -ac/\gamma, -ac/\delta, -ac/\lambda; q)_\infty}{c(q, -\alpha/c, -\beta/c, -\gamma/c, -\delta/c, -\lambda/c; q)_\infty} \\ &\cdot \frac{(q^2/a, ac^2/\alpha\beta, ac^2/\alpha\gamma, ac^2/\beta\gamma; q)_n}{(-cq/\alpha, -cq/\beta, -cq/\gamma, -a^2c^3/\alpha\beta\gamma q; q)_n}, \end{aligned}$$

where $\mathrm{Re}(c, \alpha, \beta, \gamma, \delta, \lambda) > 0$, $a^3c^5 = -\alpha\beta\gamma\delta\lambda q^2$, $ac = -\lambda q^{n+1}$, and $n = 0, 1, \ldots$.

Corresponding to (3.17), by applying (2.10) to (4.3) we obtain that

$$\tag{4.6} \begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(-iact, idqt; q)_\infty}{(-ict, idt; q)_\infty} (ia_1t; q)_{m_1} \cdots (ia_rt; q)_{m_r} dt \\ &= \frac{(-ac/d; q)_\infty}{(c+d)(-cq/d; q)_\infty} (a_1/d; q)_{m_1} \cdots (a_r/d; q)_{m_r}, \end{aligned}$$

where $|aq^{-(m_1+\cdots+m_r}| < 1$, $m_1, \ldots, m_r$ are nonnegative integers and $\mathrm{Re}(c, d, a_1, \ldots, a_r) > 0$.

By proceeding as in the proof of (4.3) with semicircles which do not pass through the poles of $1/(-ic_1, \ldots, -ic_s; q)_\infty$, we find that

$$\tag{4.7} \begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(-ia_1t, \ldots, -ia_st, ib_1t, \ldots, ib_rt; q)_\infty}{(-ic_1t, \ldots, -ic_st, id_1t, \ldots, id_rt; q)_\infty} dt \\ &= \frac{(a_1/c_1, \ldots, a_s/c_1, -b_1/c_1, \ldots, -b_r/c_1; q)_\infty}{c_1(q, c_2/c_1, \ldots, c_s/c_1, -d_1/c_1, \ldots, -d_r/c_1; q)_\infty} \\ &\cdot {}_{r+s}\phi_{r+s-1}\left[\begin{matrix} c_1q/a_1, \ldots, c_1q/a_s, -c_1q/b_1, \ldots, -c_1q/b_r \\ c_1q/c_2, \ldots, c_1q/c_s, -c_1q/d_1, \ldots, -c_1q/d_r \end{matrix}; q, \rho\right] \\ &+ \ \mathrm{idem}(c_1; c_2, \ldots, c_3). \end{aligned}$$

provided $\mathrm{Re}(c_1, \ldots, c_s, d_1, \ldots, d_r) > 0$, the integrand has only simple poles and $|\rho| < 1$, where

$$\tag{4.8} \rho = \frac{a_1a_2\cdots a_sb_1b_2\cdots b_r}{qc_1c_2\cdots c_sd_1d_2\cdots d_r}.$$

The symbol "idem$(c_1; c_2, \ldots, c_s)$" after an expression stands for the sum of the $s-1$ expressions obtained from the preceding expression by interchanging c_1 with each c_k for $k = 2, 3, \ldots, s$.

In particular, it follows from the $s=2$ case of (4.7) that

$$
\begin{aligned}
(4.9)\qquad & \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(-iact,-iac^2t/f,iac^2t/\alpha,iac^2t/\beta,iac^2t/\gamma,iac^2t/\delta;q)_\infty}{(-ict,-ift,i\alpha t,i\beta t,i\gamma t,i\delta t;q)_\infty}\left(1+\frac{ac^2t^2}{q}\right)dt\\
&=\frac{(a/q,ac/f,-ac/\alpha,-ac/\beta,-ac/\gamma,-ac/\delta;q)_\infty}{c(q,f/c,-\alpha/c,-\beta/c,-\gamma/c,-\delta/c;q)_\infty}\\
&\cdot\left\{{}_8\phi_7\left[\begin{matrix}q/a,q(q/a)^{\frac12},-q(q/a)^{\frac12},fq/ac,-\alpha q/ac,-\beta q/ac,-\gamma q/ac,-\delta q/ac\\ (q/a)^{\frac12},-(q/a)^{\frac12},cq/f,-cq/\alpha,-cq/\beta,-cq/\gamma,-cq/\delta\end{matrix};q,q\right]\right.\\
&-\frac{(ac^2/f^2q,-ac^2/f\alpha,-ac^2/f\beta,-ac^2/f\gamma,-ac^2/f\delta;q)_\infty}{(cq/f,-\alpha/f,-\beta/f,-\gamma/f,-\delta/f;q)_\infty}\\
&\cdot\frac{(fq/c,-\alpha/c,-\beta/c,-\gamma/c,-\delta/c;q)_\infty}{(a/q,-ac/\alpha,-ac/\beta,-ac/\gamma,-ac/\delta;q)_\infty}\\
&\cdot\left.{}_8\phi_7\left[\begin{matrix}\lambda,q\lambda^{\frac12},-q\lambda^{\frac12},fq/ac,-f\alpha q/ac^2,-f\beta q/ac^2,-f\gamma q/ac^2,-f\delta q/ac^2\\ \lambda^{\frac12},-\lambda^{\frac12},fq/c,-fq/\alpha,-fq/\beta,-fq/\gamma,-fq/\delta\end{matrix};q,q\right]\right\},
\end{aligned}
$$

provided $a^3c^5=f\alpha\beta\gamma\delta q^2$, $\lambda=f^2q/ac^2$, $\mathrm{Re}(c,f,\alpha,\beta,\gamma,\delta)>0$ and the integrand has only simple poles. Unfortunately, unlike in (3.10), Bailey's summation formula (2.9) for the sum of two ${}_8\phi_7$ series cannot be applied to the sum in (4.9) because of the form of the coefficient of the second ${}_8\phi_7$ in (4.9). However, if we add the restriction that $ac=fq^{n+1}$, then we can use (2.7) to sum both of the ${}_8\phi_7$ series in (4.9) and obtain the formula

$$
\begin{aligned}
(4.10)\qquad & \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{(-iact,-iac^2t/f,iac^2t/\alpha,iac^2t/\beta;q)_\infty}{(-ict,-ift,i\alpha t,i\beta t;q)_\infty}\\
&\cdot\frac{(iac^2t/\gamma,iac^2t/\delta;q)_\infty}{(i\gamma t,i\delta t;q)_\infty}(1+ac^2t^2/q)dt\\
&=\frac{(a/q,ac/f,-ac/\alpha,-ac/\beta,-ac/\gamma,-ac/\delta;q)_\infty}{c(q,f/c,-\alpha/c,-\beta/c,-\gamma/c,-\delta/c;q)_\infty}\\
&\cdot\frac{(q^2/a,ac^2/\alpha\beta,ac^2/\alpha\gamma,ac^2/\beta\gamma;q)_n}{(-cq/\alpha,-cq/\beta,-cq/\gamma,-a^2c^3/\alpha\beta\gamma q;q)_n}\\
&+\frac{(ac^2/f^2q,ac/f,-ac^2/f\alpha,-ac^2/f\beta,-ac^2/f\gamma,-ac^2/f\delta;q)_\infty}{f(q,c/f,-\alpha/f,-\beta/f,-\gamma/f,-\delta/f;q)_\infty}\\
&\cdot\frac{(f^2q^2/ac^2,ac^2/\alpha\beta,ac^2/\alpha\gamma,ac^2/\beta\gamma;q)_n}{(-fq/\alpha,-fq/\beta,-fq/\gamma,-a^2c^4/f\alpha\beta\gamma q;q)_n},
\end{aligned}
$$

provided $ac=fq^{n+1}$, $a^3c^5=f\alpha\beta\gamma\delta q^2$, $\mathrm{Re}(c,f,\alpha,\beta,\gamma,\delta)>0$, the integrand has only simple poles, and $n=0,1,\ldots$.

Note that if we also apply the residue theorem to the integral in (4.7) by using the poles in the lower half plane and set the result equal to the right side of (4.7), we obtain the transformation formula

$$
\begin{aligned}
(4.11)\qquad & \frac{(a_1/c_1,\ldots,a_s/c_1,-b_1/c_1,\ldots,-b_r/c_1;q)_\infty}{c_1(q,c_2/c_1,\ldots,c_s/c_1,-d_1/c_1,\ldots,-d_r/c_1;q)_\infty} \\
& \cdot {}_{r+s}\phi_{r+s-1}\left[\begin{matrix} c_1q/a_1,\ldots,c_1q/a_s,-c_1q/b_1,\ldots,-c_1q/b_r \\ c_1q/c_2,\ldots,c_1q/c_s,-c_1q/d_1,\ldots,-c_1q/d_r \end{matrix};q,\rho\right] \\
& + \mathrm{idem}(c_1;c_2,\ldots,c_s) \\
& = \frac{(b_1/d_1,\ldots,b_r/d_1,-a_1/d_1,\ldots,-a_s/d_1;q)_\infty}{d_1(q,d_2/d_1,\ldots,d_r/d_1,-c_1/d_1,\ldots,-c_s/d_1;q)_\infty} \\
& \cdot {}_{r+s}\phi_{r+s-1}\left[\begin{matrix} d_1q/b_1,\ldots,d_1q/b_r,-d_1q/a_1,\ldots,-d_1q/a_s \\ d_1q/d_2,\ldots,d_1q/d_r,-d_1q/c_1,\ldots,-d_1q/c_s \end{matrix};q,\rho\right] \\
& + \mathrm{idem}(d_1;d_2,\ldots,d_r),
\end{aligned}
$$

provided the denominators of the coefficients of the ϕ series and of the terms in the series do not vanish and $|\rho| < 1$, where ρ is as defined in (4.8). The transformation formula which follows by applying this procedure to integrals of the type in (3.18) is given in Slater [30, (5.2.16)]

5. Remarks. In 1944 Selberg [29] extended Euler's beta integral on $[0,1]$ to the multivariable integral

$$
\begin{aligned}
(5.1)\qquad & \int_0^1\cdots\int_0^1 \prod_{j=1}^n t_j^{x-1}(1-t_j)^{y-1} \prod_{1\le j<k\le n} |t_j-t_k|^{2z}\,dt_1\cdots dt_n \\
& = \prod_{j=1}^n \frac{\Gamma(x+(j-1)z)\Gamma(y+(j-1)z)\Gamma(jz+1)}{\Gamma(x+y+(n+j-2)z)\Gamma(z+1)},
\end{aligned}
$$

where $\mathrm{Re}(x) > 0$, $\mathrm{Re}(y) > 0$, and $\mathrm{Re}(z) > -\min(1/n, \mathrm{Re}(x)/(n-1), \mathrm{Re}(y)/(n-1))$. Askey [4] conjectured a q-extension of Selberg's integral which was recently proved by Habsieger [21] and Kadell [25]. Other q-extensions of (5.1) were conjectured in Askey [4] and Rahman [28]. For a brief survey of applications of Selberg's integral to root systems and recent extensions and conjectured extensions of it, see Zeilberger [34]. It would be of interest if multivariable extensions of Cauchy's and Barnes' beta integrals and their q-extensions could be derived.

Recently the author [18] derived the bibasic indefinite summation formula

$$
\begin{aligned}
(5.2)\qquad & \sum_{k=0}^n \frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)} \frac{(a,b;p)_k(c,a/bc;q)_k}{(q,aq/b;q)_k(ap/c,bcp;p)_k} q^k \\
& = \frac{(ap,bp;p)_n(cq,aq/bc;q)_n}{(q,aq/b;q)_n(ap/c,bcp;p)_n},
\end{aligned}
$$

where p and q are independent bases and $n = 0, 1, \ldots$, and used it to derive a bibasic extension of Euler's transformation formula, a bibasic expansion formula, and some quadratic, cubic and quartic summation formulas. This suggests the problem of finding bibasic functions for which integrals of the types in (3.18) and (4.2) can be evaluated.

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