THE H-PRINCIPLE, LECTURE 10: CLASSIFYING IMMERSIONS OF SPHERES, AFTER SMALE

J. FRANCIS, NOTES BY A. BEAUDRY

We will end our treatment of immersion theory by classifying immersions of spheres. This was done by Smale before the Smale-Hirsch theorem. The theorem implies the weak homotopy equivalence

$$\mathrm{Imm}(S^n, \mathbb{R}^{n+k}) \simeq \mathrm{Imm}^{\mathrm{f}}(S^n, \mathbb{R}^{n+k}) \simeq \mathrm{Map}^{\mathrm{inj}}_{\mathrm{Vect}(S^n)}(T_{S^n}, \underline{\mathbb{R}}^{n+k}).$$

If S^n is parallelizable, this is the same as a map from S^n to an injective linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^{n+k}$, i.e., an n-frame in $\mathbb{R}^{n+k}$. Hence, in this case,

$$\mathrm{Imm}(S^n, \mathbb{R}^{n+k}) \simeq \pi_n V_n(\mathbb{R}^{n+k}) \simeq \pi_n O(n+k)/O(k).$$

In favorable cases, we know how to compute these homotopy groups, but the sphere is parallelizable only in dimensions $n = 0$, 1, 3, and 7. We would like to understand in other dimensions.

We know the answer does not change for a fattened sphere. Since $S^n \times D^1$ is parallelizable, then

$$\mathrm{Imm}^{\mathrm{f}}(S^n \times D^1, \mathbb{R}^{n+k}) \simeq \mathrm{Map}(S^n, V_{n+1}(\mathbb{R}^{n+k})).$$

Our hope would be that this also holds for S^n, at least on π_0.

In homotopy theory, it often is easier to fix a base point, so our first simplification is to work with based immersions.

Definition 0.1. Let $x \in S^n$ be a fixed based point. Define $\mathrm{Imm}_*(S^n, \mathbb{R}^{n+k})$ by the following pullback diagram,

$$\begin{tikzcd}
\mathrm{Imm}_*(S^n, \mathbb{R}^{n+k}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & \mathrm{Imm}(S^n, \mathbb{R}^{n+k}) \arrow[d] \\
* \arrow[r] & V_n(T_{\mathbb{R}^{n+k}})
\end{tikzcd}$$

Given any f and g in $\mathrm{Imm}_*(S^n, \mathbb{R}^{n+k})$ we can deform g by homotopy so that f and g agree on some D^n_x neighborhood of x. Define $[f - g]$ in the nth homotopy group $\pi_n V_n(\mathbb{R}^{n+k})$ by gluing together $f|_{S^n - D^n_x}$ and $g|_{S^n - D^n_x}$. This defines a map from S^n to $V_n(\mathbb{R}^{n+k})$ which only depends on the homotopy classes of f and g. Fixing f, we get the following theorem.

Theorem 0.2. (Smale) *There is a non-canonical isomorphism*

$$\pi_0 \mathrm{Imm}_*(S^n, \mathbb{R}^{n+k}) \simeq \pi_n V_n(\mathbb{R}^{n+k}).$$

We will prove the following corollary.

Corollary 0.3. *For $k > 1$,*

$$\pi_0 \mathrm{Imm}(S^n, \mathbb{R}^{n+k}) \simeq \pi_n V_n(\mathbb{R}^{n+k}).$$

Proof. We use the long exact sequence on homotopy groups induced by the fiber sequence,

$$\mathrm{Imm}_*(S^n, \mathbb{R}^{n+k}) \to \mathrm{Imm}(S^n, \mathbb{R}^{n+k}) \to V_n(\mathbb{R}^{n+k}).$$

Since $V_n(\mathbb{R}^{n+k})$ is homotopy equivalent to $O(n+k)/O(k)$, it is simple to compute that $\pi_0 V_n(\mathbb{R}^{n+k})$ is zero for $k > 0$. For $k = 1$, $O(n+1)/O(1)$ is $SO(n+1)$, but if $k > 1$ the connectivity of

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the map $O(k) \to O(n+k)$ implies that $\pi_1 V_n(\mathbb{R}^{n+k})$ is zero. Putting these together we get that $\pi_0 \operatorname{Imm}_*(S^n, \mathbb{R}^{n+k}) \to \pi_0 \operatorname{Imm}(S^n, \mathbb{R}^{n+k})$ is an isomorphism. $\square$

When $k = 1$, the map is surjective. With some extra information about the long exact sequence we can hope to also understand this case. We have reduced the problem of computing immersions to that of computing based immersions, which is the same as the group $\pi_n V_n(\mathbb{R}^{n+k})$.

The first case is when $n = 1$ and $k = 1$. Here $V_1(\mathbb{R}^2)$ is homotopy equivalent to $SO(2)$, hence $\pi_0 \operatorname{Imm}(S^1, \mathbb{R}^2)$ is just $\mathbb{Z}$. These immersions are classified by winding numbers. One interesting consequence is that it is not possible to turn the circle inside out in $\mathbb{R}^2$.

The second case is when $n = 2$ and $k = 1$. Since $\pi_2 SO(3)$ is zero, we conclude that $\pi_0 \operatorname{Imm}_*(S^2, \mathbb{R}^3)$ is zero, so that all immersions of the 2-sphere in $\mathbb{R}^3$ are equivalent. In particular, you can turn the sphere inside out in $\mathbb{R}^3$.

Observe that as k increases, the map $O(k) \to O(n+k)$ becomes highly connected so that the same holds for $V_n(\mathbb{R}^{n+k}) \simeq O(n+k)/O(k)$. More precisely, if $k > n$, then $\pi_n V_n(\mathbb{R}^{n+k})$ is zero and all immersions of S^n in $\mathbb{R}^{n+k}$ are equivalent.

An interesting question is that of reversing S^n in $\mathbb{R}^{n+1}$. To answer this, we need to compute $\pi_0 \operatorname{Imm}_*(S^n, \mathbb{R}^{n+1}) \simeq \pi_n SO(n+1)$. If this group is zero, then the answer to the inside out question is positive. We can study $SO(n+1)$ using Bott periodicity and the fiber sequence

$$SO(n+1) \to SO(n+2) \to S^{n+1}.$$

We have that $\pi_i SO(n+1) \simeq \pi SO$ when $i \leq n-1$. Here $\pi_i SO$ denotes the stable homotopy groups of the spectrum SO. From the long exact sequence on homotopy groups and the fact that $\pi_{n+1} S^{n+1} = \mathbb{Z}$ and $\pi_n S^{n+1} = 0$, we get an exact sequence,

$$\mathbb{Z} \to \pi_n SO(n+1) \to \pi_n SO \to 0.$$

By Bott periodicity, the groups $\pi_n SO$ are non-zero only when $i \equiv 0, 1, 3$ and 7 modulo 8. In these cases $\pi_n SO(n+1)$ surjects onto a non-zero group, and hence is non-trivial. Otherwise, it receives a surjective map from $\mathbb{Z}$. So it is not apparent from these calculuations whether or not you can everse the n-sphere in $\mathbb{R}^{n+1}$ for higher n. For example, we know that $\pi_2 SO(3)$ is zero but $\pi_3 SO(4) \simeq \mathbb{Z} \times \mathbb{Z}$. A good exercise is to compute $\pi_4 SO(5)$.

This ends our treatment of immersion theory.

References

[1] Smale, Stephen. The classification of immersions of spheres in Euclidean spaces. Ann. Math. 69 (1959), 327-344.