THE H-PRINCIPLE, LECTURE 13: CLASSIFYING FOLIATIONS

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Recall the following definitions from the last lecture.

Definition 0.1. Γ_q is the topological groupoid of germs of local diffeomorphisms of $\mathbb{R}^q$. Its space of objects is $\mathbb{R}^q$, and the space of morphisms Γ_q consists of germs of diffeomorphisms, $(x, U) \to (y, V)$, for all $x, y \in \mathbb{R}^q$. The topology of Γ_q is induced from that of $\mathbb{R}^q \times \mathrm{Diff}(\mathbb{R}^q)^\delta$, where $\mathrm{Diff}(\mathbb{R}^q)^\delta$ has the discrete topology.

Definition 0.2. *A Haefliger Γ_q-structure* on M consists of $\{U_\alpha \subset M\}$ a cover of M, and maps $\varphi_{\alpha\beta} \colon U_{\alpha\beta} \to \Gamma_q$ satisfying the cocycle condition.

Let us further introduce the following.

Definition 0.3. Γ_q*-foliation* of M is a Haefliger Γ_q-structure on M for which each of the composites

$$\mathrm{proj_s} \circ \varphi_{\alpha\beta} \colon U_{\alpha\beta} \longrightarrow \Gamma_q \longrightarrow \mathbb{R}^q,$$

where $\mathrm{proj_s}$ is the projection taking the sources of the morphisms, are submersions. (For this, it is in fact enough that $\mathrm{proj_s} \circ \varphi_{\alpha\alpha}$ are submersions.)

Note that a foliation of M of codimension-q gives rise to a structure of Γ_q-foliation. Indeed, the transition functions for the foliated charts $U_\alpha \xrightarrow{\cong} \mathbb{R}^q \times \mathbb{R}^{n-q}$ descends to local diffeomorphisms of $\mathbb{R}^q$. One can

For classifying foliations, we want to encode the data of Γ_q-foliations in terms of Γ_q-structures. The value of this will be from the fact that Γ_q-structures are determined by homotopy theory. For example, we have $H^1(M, \Gamma_q) \cong [M, B\Gamma_q]$, where $B\Gamma_q$ is the classifying space of the topological groupoid Γ_q, constructed e.g. as the suitable quotient

$$\coprod_i (\Gamma_q)_i \times \Delta^i / \sim,$$

where $(\Gamma_q)_i$ is the space of length i chain of composable morphisms, namely

$$\Gamma_q \times_{\mathbb{R}^q} \Gamma_q \times_{\mathbb{R}^q} \cdots \times_{\mathbb{R}^q} \Gamma_q.$$

There is one basic difference between Γ_q-structures and Γ_q-foliations. Observe first that Γ_q-structures pull back. That is, for any map $f \colon M \to E$ and $\mathfrak{F} = \{U_\alpha \subset E, \varphi_{\alpha\beta}\}$ a Γ_q-structure on E, we obtain a Γ_q-structure

$$f^{-1}\mathfrak{F} = \{f^{-1}U_\alpha, f^{-1}U_{\alpha\beta} \to U_{\alpha\beta} \xrightarrow{\varphi_{\alpha\beta}} \Gamma_q\}$$

on M.

However, Γ_q-foliations do not pull back for arbitrary maps. For example, foliate $\mathbb{R}^2$ by lines and consider a submanifold which looks as in Figure 1. Then the restriction of the Γ_q-foliation to the submanifold is not a Γ_q-foliation.

You don't genuinely understand a structure until you've pinned down its functoriality. This begs the question:

Question 0.4. For what class of maps can you pull back a foliation?

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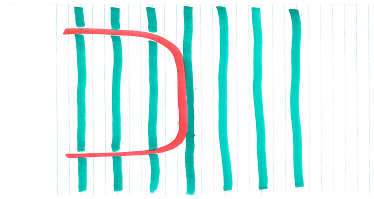

FIGURE 1. Green: leaves. Red: submanifold

Answers will be, in increasing order of generality:

- Open embeddings;
- Submersions;
- Maps transversal to the foliation.

The last notion is defined as follows.

Definition 0.5 (Intuitive (i.e.,, in terms of leaves))**.** A map $f\colon M \to E$ is *transversal* to $\mathfrak{F}$, a foliation of E, if for every leaf $E_\alpha \subset E$, the map f is transversal to E_α: $f \pitchfork E_\alpha$. I.e., for every $x \in f^{-1}(E_\alpha)$, the composite

$$T_{f(x)}E_\alpha \oplus T_x M \longrightarrow T_{f(x)}E$$

is surjective, or equivalently, the composite

$$T_x M \longrightarrow T_{f(x)}E \longrightarrow N_{\mathfrak{F}}|_{f(x)}$$

is surjective.

Here's an equivalent definition.

Definition 0.6. The map $f\colon M \to E$ is transversal to $\mathfrak{F}$, a Γ_q-foliation of E, if the composite map

$$f^{-1}U_\alpha \to U_\alpha \xrightarrow{\mathrm{proj}_s \varphi_{\alpha\alpha}} \mathbb{R}^q$$

is a submersion for each U_α.

Exercise: Check that these definitions are equivalent.

Definition 0.7. The space of maps $M \to E$ transversal to $\mathfrak{F}$, a Γ_q-foliation of E, denoted

$$\mathrm{Map}_{\pitchfork\mathfrak{F}}(M, E)$$

is the subspace of smooth maps $\mathrm{Map}^{\mathrm{sm}}(M, E)$ formed by all f such that the composite

$$T_M \xrightarrow{df} T_E \longrightarrow N_{\mathfrak{F}}$$

is surjective on every fiber.

Observe that this last condition is a differential relation! That is, we can choose $\mathcal{R} \subset (M \times E)^{(1)}$ (bundle of 1-jets on M) to be the subspace whose fiber at a point $x \in M$ consists of those linear maps $T_x M \to T_y E$ which $T_x M$ surjects onto $N_{\mathfrak{F}}|_y$, and then we have the equalities

$$\mathrm{Map}_{\pitchfork\mathfrak{F}}(M, E) = \mathrm{Sol}_{\mathcal{R}}(M),$$

$$\mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(T_M, T_E) := \Gamma_M(\mathcal{R}).$$

Theorem 0.8 (Gromov, Phillips)**.** *If M is an open n-manifold, then the map*

$$\mathrm{Map}_{\pitchfork\mathfrak{F}}(M, E) \longrightarrow \mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(T_M, N_{\mathfrak{F}})$$

is a weak homotopy equivalence.

Remark 0.9. If the foliation $\mathfrak{F}$ is that by points of E, then $\mathrm{Map}_{\pitchfork\mathfrak{F}}(M, E) = \mathrm{Subm}(M, E)$ is the space of submersions, $\mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(T_M, N_{N_{\mathfrak{F}}}) = \mathrm{Subm}^{\mathrm{f}}(M, E)$ is the space of formal submersions (since $N_{\mathfrak{F}} = T_E$ in this case), and the theorem above reduces to the previously seen case of the Gromov-Phillips theorem for submersions out of an open manifold.

Proof. We apply Gromov's theorem that h-principle holds for open, diffeomorphism-invariant, differential relations on open manifolds (see Lecture 11). So all we need do is verify openness and invariance of $\mathcal{R}$.

Openness of $\mathcal{R}$ in the space of all 1-jets is obvious: This is a parametrized version of observation that $\mathrm{Hom}^{\mathrm{surj}}(V, W) \subset \mathrm{Hom}(V, W)$ is an open subspace for any vector spaces V, W over $\mathbb{R}$.

Diffeomorphism invariance, that $\mathrm{Diff}(M)$ preserves $\mathcal{R}$, is also obvious, since precomposition of a map with a diffeomorphism doesn't change the surjectivity condition above. $\square$

Observe that differentiation of local diffeomorphisms defines a functor

$$d\colon (\mathbb{R}^q, \Gamma_q) \longrightarrow (*, GL_q).$$

Using the map Bd induced from this functor on the classifying spaces, we can pull back the universal dimension q vector bundle over BGL_q to over $B\Gamma_q$. Let us call this vector bundle N_{Γ_q}.

Given a Γ_q-foliation $\mathfrak{F}$, observe that the submersion $U_\alpha \to \mathbb{R}^q$ identifies $\varphi_{\alpha\alpha}^{-1} T_{\mathbb{R}^q}$ with $N_{\mathfrak{F}}|_{U_\alpha}$. Therefore, if we identify $\mathfrak{F}$ with the map $M \to B\Gamma_q$ classifying it as a Γ_q-structure, then $N_{\mathfrak{F}}$ will be $\mathfrak{F}^* N_{\Gamma_q}$.

Theorem 0.10 (Haefliger). *Let M be an open n-manifold. Then $\mathrm{Fol}_q(M)/_\sim$, the set of codimension-q foliations of M modulo integrable homotopy, is naturally in bijection with the components of the space of surjective bundle maps $\mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(T_M, N_{\Gamma_q})$:*

$$\mathrm{Fol}_q(M)/_\sim \cong \pi_0 \, \mathrm{Map}^{\mathrm{surj}}_{\mathrm{Vect}}(T_M, N_{\Gamma_q}).$$

These sets are also in bijection with π_0 of the space of all lifts

$$\begin{tikzcd}
& B\Gamma_q \times BGL_{n-q} \arrow[d] \\
& BGL_q \times BGL_{n-q} \arrow[d] \\
M \arrow[r, "T_M"] \arrow[uur, dashed] & BGL_n.
\end{tikzcd}$$

The proof will be given next time.

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