THE H-PRINCIPLE, LECTURE 17: THE SHEAF OF CONFIGURATION SPACES AND THE SCANNING MAP

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1. Configuration spaces and mapping spaces

Our aim in the next several lectures is to prove the following theorem of Dusa McDuff. We will state it first, then define the terms.

Theorem 1.1. [1] *Let X be a pointed space and M be an n-manifold. Then the scanning map*

$$\mathrm{Conf}_X(M) \longrightarrow \Gamma_{\mathrm{c}}(T_M^\infty \wedge_M X)$$

is a weak homotopy equivalence if either

- *M is open and compact, or*
- *X is connected.*

We now define the constituent terms:

- $\mathrm{Conf}_X(M) := \mathrm{Conf}_X(M, \partial M)$ is the configuration space of points in M labeled by X, and with annihilation of points in the boundary of M. (M is not assumed to have boundary or be compact.) See Lecture 16.
- T_M^∞ is the fiberwise 1-point compactification of the tangent bundle of M, i.e., the bundle of pointed n-spheres over M formed by adding a point at infinity in each space tangent space $T_{M,x}$, $x \in M$. Sections of T_M^∞ can be thought of possibly infinite vector fields.[2]
- $T_M^\infty \wedge_M X$ is the fiberwise smash product over M. The fiber over a point x is $T_{M,x}^\infty \wedge X \cong \Sigma^n X$.
- Γ_{c} denotes compactly supported sections, i.e., sections which are constant basepoint (in this case, going to the point at infinity) outside a compact subspace of M, and equipped with the compact-open topology. (If M is compact then, of course, we have $\Gamma_{\mathrm{c}} = \Gamma$.)

This leaves only to define the scanning map, which will come later in this lecture. Before doing so, let us first observe some consequences of the above theorem:

Corollary 1.2. *If M is a parallelizable manifold, then there is a homotopy equivalence*

$$\mathrm{Conf}_X(M) \longrightarrow \mathrm{Map}_{\mathrm{c}}(M, \Sigma^n X)$$

if either M is open and compact or X is connected.

Proof. A framing $T_M \cong \underline{\mathbb{R}}^n$ gives a homeomorphism of pointed spaces $T_M^\infty \wedge_M X \cong M \times \Sigma^n X$ over M.

□

Example 1.3. Consider $M = \mathbb{R}$. We saw previously that there was a natural homotopy equivalence $\mathrm{Conf}_X(\mathbb{R}) \simeq JX$, to the James construction. By identifying the open interval and $\mathbb{R}$, $(0,1) \cong \mathbb{R}$, there is further an equivalence $\mathrm{Map}_{\mathrm{c}}(\mathbb{R}, \Sigma X) \simeq \Omega\Sigma X$ with the based loop space of ΣX. We therefore recover James' original result, that there is a homotopy equivalence $JX \simeq \Omega\Sigma X$, for X a connected, pointed space. Note the necessity of the hypothesis that X is connected: If X has

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Date: Lecture February 21, 2010. Last edited on February –, 2010.

[1] This theorem and its proof are essentially due to McDuff, though the formulation with a space of labels seems to be first formulated by Bödigheimer. The proof uses ideas of Segal and Gromov.

[2] Note that this is *different* from the Thom space of T_M. The Thom space $\mathrm{Th}(T_M)$ is a further quotient of T_M^∞ obtained by collapsing all the different ∞'s in all the fibers to a single point.

multiple components, then $\pi_0 JX$ is a monoid which is not a group, while $\pi_0\Omega\Sigma X \cong \pi_1\Sigma X$ is a group, and they will never be homotopy equivalent.

Example 1.4. Consider now $M = \mathbb{R}^n$. Then $\mathrm{Conf}_X(\mathbb{R}^n)$ is homotopy equivalent to the free $\mathcal{E}_n$-algebra $\mathrm{Free}_{\mathcal{E}_n}(X,*)$ generated by the pointed space X. The theorem then implies the homotopy equivalence $\mathrm{Free}_{\mathcal{E}_n}(X,*) \simeq \mathrm{Conf}_X(\mathbb{R}^n) \simeq \Omega^n\Sigma^n X$, a result originally due to Segal and May.

2. Sheaves on manifolds, revisited

Until this point, when we considered a sheaf, such as sheaf of immersions with fixed target $\mathrm{Imm}(-,N)$, we considered it as a sheaf on opens of a single manifold M^n. However, all of the examples we have considered (such as $\mathrm{Imm}(-,N)$, $\mathrm{Map}^{\mathrm{sm}}(-,N)$, $\mathrm{Subm}(-,N)$, $\mathrm{Fol}_q(-)$, $\mathrm{Conf}_X(-)$) can be naturally considered as sheaves on all n-manifolds at once. At this point, it becomes beneficial to pursue this line of thinking.

Definition 2.1. Mfld_n is the topological category of smooth, **compact** n-dimensional manifolds with **embeddings** as morphisms. Namely, $\mathrm{Map}_{\mathrm{Mfld}_n}(M,N) = \mathrm{Emb}(M,N)$.

Definition 2.2. Let M be a compact n-manifold. The collection of embeddings $\{f_\alpha : U_\alpha \hookrightarrow U | \alpha \in J\}$ is a cover by **compact** n-manifolds if:

- The map $\coprod_J U_\alpha \to U$ is surjective;
- For any subset $J_0 \subset J$, the intersection $\bigcap_{J_0} f_\alpha(U_\alpha) \subset U$ is a closed, embedded n-manifold.

Remark 2.3. For instance, we do not allow the two closed hemispheres of S^2 to form a cover, because their intersection is the equator S^1, which is codimension 1. However, if we stretch the hemispheres to overlap in a band $S^1 \times [-\epsilon, \epsilon]$, then this becomes a cover in the sense of the above definition.

Definition 2.4. $\mathrm{Shv}(\mathrm{Mfld}_n)$ is the full subcategory of continuous functors $\mathrm{Fun}(\mathrm{Mfld}_n^{\mathrm{op}}, \mathrm{Spaces})$ consisting of those presheaves $\mathcal{F}$ for which the natural map

$$\mathcal{F}(U) \longrightarrow \lim \left(\prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta) \right)$$

is a homeomorphism.[3] In particular, this all such sheaves $\mathcal{F}$ have the property that $\mathcal{F}(i) : \mathcal{F}(V) \to \mathcal{F}(U)$ is a weak homotopy equivalence for every isotopy equivalence $i : U \hookrightarrow V$.[4]

Recall that an embedding $f : U \to V$ is an *isotopy equivalence* if it is isotopic to a diffeomorphism. I.e., there exists a smooth family of embeddings $f_t : [0,1] \times U \to V$ with $f_0 = f$ and f_1 is a diffeomorphism. Note that the space $\mathrm{Diff}(V)$ acts of $\mathcal{F}(V)$, for $\mathcal{F} \in \mathrm{Shv}(\mathrm{Mfld}_n)$.

Remark 2.5.

3. The scanning map

Let $\mathcal{F}$ be a sheaf on manifolds, as we've just discussed. The values $\mathcal{F}(M)$ can be approximated by the sheaf of sections $\Gamma_\mathcal{F}$ of a bundle $E_\mathcal{F}(M)$ on each M, which we now construct: Let

$$E_\mathcal{F}(M) := \mathrm{Frame}(T_M) \times_{GL_n} \mathcal{F}(D^n)$$

be the diagonal quotient by GL_n of the principal GL_n bundle of n-frames of M and the value of $\mathcal{F}$ on the standard n-disk. Choosing a Riemannian metric on M, the fibers of the bundle $E_\mathcal{F}(M)$ can

[3]You may notice that I granted my own previous wish to work with sheaves defined on compact manifolds, rather than open subspaces of a manifold.

[4]For instance, this excludes the sheaf of smooth structures Sm on the category of 4-manifolds.

be continuously identified with the spaces $\mathcal{F}(\mathrm{Disk}(T_{M,x}))$, $\mathcal{F}$ applied to the unit disk bundle of the tangent space of M at x:

$$\begin{array}{ccc} \mathcal{F}(\mathrm{Disk}(T_{M,x})) \cong \mathrm{Fib}_x & \longrightarrow & E_{\mathcal{F}}(M) \\ \downarrow & & \downarrow \\ \{x\} & \longrightarrow & M \end{array}$$

We define $\Gamma_{\mathcal{F}}$ as the sheaf of sections on M of $E_{\mathcal{F}}(M)$. We now construct the scanning map:

$$\mathcal{F}(M) \xrightarrow{\text{scan}} \Gamma_{\mathcal{F}}(M)$$

For each $f \in \mathcal{F}(M)$, we construct a section $\mathrm{scan}(f)$ of the bundle: $\mathrm{scan}(f) : M \to E_{\mathcal{F}}(M)$, assigning to each point $x \in M$ an element of the fiber of $E_{\mathcal{F}}(M)$ over x, which we can identify with $\mathcal{F}(\mathrm{Disk}(T_{M,x}))$. Using the exponential map $\exp_x : \mathrm{Disk}(T_{M,x}) \hookrightarrow M$, we have an induced map $\mathcal{F}(\exp_x) : \mathcal{F}(M) \to \mathcal{F}(\mathrm{Disk}(T_{M,x}))$, and we define the value of $\mathrm{scan}(f)$ at x to be

$$\mathrm{scan}(f)(x) := \mathcal{F}(\exp_x)(f)$$

which varies continuously in x, and hence defines a section of $E_{\mathcal{F}}(M)$.

4. The h-principle for sheaves

Definition 4.1. For $\mathcal{F}$ a sheaf on manifolds, as above, $\mathcal{F}$ adheres to the h-principle on M if the scanning map $\mathcal{F}(U) \to \Gamma_{\mathcal{F}}(U)$ is a weak homotopy equivalence for every $U \subset M$.

Proposition 4.2. *For $\mathcal{R}$ an open, diffeomorphism invariant, differential relation on M, then $\mathcal{R}$ adheres to the h-principle (for differential relations) if and only if the sheaf of solutions $\mathrm{Sol}_{\mathcal{R}}$ adheres to the principle (for sheaves).*

Proof. $\Gamma_{\mathcal{R}} \simeq \Gamma_{\mathrm{Sol}_{\mathcal{R}}}$.

□

Next time we will prove McDuff's theorem, which will be seen as the statement that the sheaf Conf_X on adheres to the h-principle on M, given the aforementioned conditions.

References

[1] Bödigheimer, C.-F. Stable splittings of mapping spaces. Algebraic topology (Seattle, Wash., 1985), 174187, Lecture Notes in Math., 1286, Springer, Berlin, 1987.

[2] Gromov, Mikhael. Partial differential relations. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 9. Springer-Verlag, Berlin, 1986. x+363 pp.

[3] McDuff, Dusa. Configuration spaces. K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), pp. 88–95. Lecture Notes in Math.,#Vol. 575, Springer, Berlin, 1977.