THE H-PRINCIPLE, LECTURE 18: THE PROOF OF MCDUFF'S THEOREM, FIRST PART

J. FRANCIS, NOTES BY C. ELLIOTT

1. McDuff's Theorem on Configuration Spaces

We're going to use the machinery of the h-principle for sheaves and the scanning map to prove Bödigheimer's generalization of a theorem of McDuff. Let's remind ourselves of what it said:

Theorem 1.1. *There exists a map*

$$\mathrm{Conf}_X(M) \to \Gamma_c(T_M^\infty \wedge_M X)$$

which is a weak homotopy equivalence if either X is connected or M is compact and open.

Here, $\mathrm{Conf}_X(M)$ is the configuration space of points of M labeled by points of X (with annihilation/creation of points on the boundary of M), and T_M^∞ is the fiberwise 1-point compactification of the tangent bundle. First we'll prove the theorem in the case $M = D^n$.

Lemma 1.2. $\mathrm{Conf}_X(D^n)$ *is homotopy equivalent to the n^{th} suspension $\Sigma^n X$ of X.*

Proof. Note that we can filter $\mathrm{Conf}_X(M)$ by the number of points of M in a configuration:

$$\cdots \subseteq \mathrm{Conf}_X(M)_{\leq i} \subseteq \mathrm{Conf}_X(M)_{\leq i+1} \subseteq \cdots .$$

So in particular

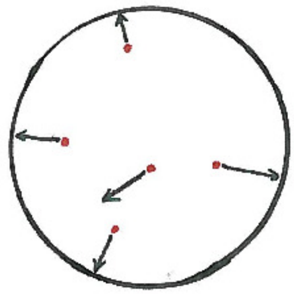
$$\begin{aligned}
\mathrm{Conf}_X(M)_{\leq 0} &= \mathrm{pt} \\
\mathrm{Conf}_X(M)_{\leq 1} &= (\mathrm{Conf}_1(M) \times_{\Sigma_1} X^1)/(\mathrm{Conf}_1(\partial M) \times X) \\
&= (D^n \times X)/(S^{n-1} \times X) \cong \Sigma^n X
\end{aligned}$$

in the case $M = D^n$. Now, let's construct a deformation retraction

$$\begin{tikzcd}
\mathrm{Conf}_X(D^n) \times \{0\} \arrow[d] \arrow[dr, "\mathrm{id}"] & \\
\mathrm{Conf}_X(D^n) \times [0,1] \arrow[r, "r_t"] & \mathrm{Conf}_X(D^n) \\
\mathrm{Conf}_X(D^n) \times \{1\} \arrow[u] \arrow[r, "\text{retract}"] & \mathrm{Conf}_X(D^n)_{\leq 1} \arrow[u]
\end{tikzcd}.$$

The idea of this construction is to dilate the disk so that all the points except maybe one go to the boundary. Define the dilation r_t for a configuration f so that all but maybe the furthest point of the configuration map to ∂D^n. One can choose a way of doing this that is continuous in f.

Date: Lecture February , 2010. Last edited on February –, 2010.



$\square$

For the rest of this lecture we assume M is *compact*. Recall that we have a bundle on M corresponding to the sheaf Conf_X, with fibers $\mathrm{Conf}_X(\mathrm{Disk}(T_xM))$,

$$\begin{tikzcd}
\mathrm{Conf}_X(\mathrm{Disk}(T_xM)) \arrow[r] \arrow[d] & E_{\mathrm{Conf}_X} \arrow[d] \\
\{x\} \arrow[r] & M.
\end{tikzcd}$$

We saw the total space of this bundle could be defined as

$$\mathrm{Frame}(T_M) \times_{GL_n} \mathrm{Conf}_X(D^n)$$

so by the lemma, there's a natural homotopy equivalence over M

$$\mathrm{Frame}(T_M) \times_{GL_n} \mathrm{Conf}_X(D^n) \xrightarrow{\sim} \mathrm{Frame}(T_M) \times_{GL_n} \Sigma^n X \cong T_M^\infty \wedge_M X.$$

We also constructed a *scanning map* for our sheaf, which allows us to define the map of the theorem as the composition

$$\mathrm{Conf}_X(M) \xrightarrow{\text{scan}} \Gamma(\mathrm{Conf}_X(M)) \to \Gamma(T_M^\infty \wedge_M X).$$

Now, let's return for our usual recipe for proving results of this form:

(1) Prove the theorem for $M = D^n$.
(2) Induct on a handle decomposition of M.

For the $M = D^n$ case, we've constructed maps

$$\begin{tikzcd}
\mathrm{Conf}_X(D^n) \arrow[r] \arrow[d, "\sim"] & \Gamma(T_{D^n}^\infty \wedge_{D^n} X) \arrow[d, "\sim"] \\
\Sigma^n X \arrow[ddr, "\sim"] & D^n \times \Sigma^n X \arrow[d, "\sim"] \\
 & \mathrm{Map}(D^n, \Sigma^n X) \arrow[d, "\sim"] \\
 & \Sigma^n X
\end{tikzcd}$$

which shows the theorem holds for D^n.

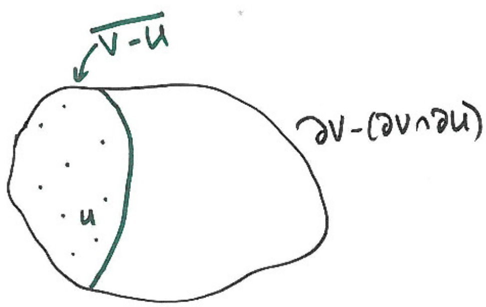
To understand the inductive step, we need to understand the restriction maps $\mathrm{Conf}_X(V) \to \mathrm{Conf}_X(U)$, where $U \hookrightarrow V$ is, say, of the form

$$S^{k-1} \times [0, \delta] \times D^{n-k} \hookrightarrow D^k \times D^{n-k}.$$

We observe that the fibers of this map look like

$$\begin{tikzcd}
\mathrm{Conf}_X(\overline{V-U}, \partial V - \partial V \cap \partial U) \arrow[r] \arrow[d] & \mathrm{Conf}_X(V) \arrow[d] \\
\{f\} \arrow[r] & \mathrm{Conf}_X(U).
\end{tikzcd}$$

Indeed, we can see this immediately from the picture



So we have a map with fibers that are all homeomorphic. Note that this doesn't imply that it is a fibration, as the following example shows:

Example 1.3. As in the case of immersion theory (in which we saw that $\mathrm{Imm}(D^n, N^n) \to \mathrm{Imm}(S^{n-1} \times [0,\delta], N^n)$ was not a Serre fibration), let's look for failure in the case

$$U = S^{n-1} \times [0,\delta] \hookrightarrow D^n = V.$$

Let $X = S^0$, or any non-connected space. So we have

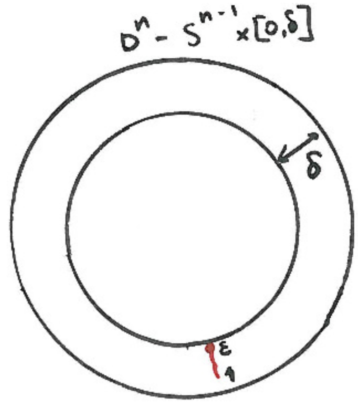
$$\begin{tikzcd}
\coprod_i (\mathrm{Conf}_i(D^n_{1-\delta})_{\Sigma_i} \arrow[r, equal] & \mathrm{Conf}(D^n_{1-\delta}, \varnothing) \arrow[r] \arrow[d] & \mathrm{Conf}(D^n) \arrow[d] \\
 & \{f\} \arrow[r] & \mathrm{Conf}(S^{n-1} \times [0,\delta]).
\end{tikzcd}$$

In particular, the fibers have infinitely many non-homeomorphic components, but the base and total space are connected, so the map cannot be a fibration. To elaborate, we know that $\mathrm{Conf}(D^n) \cong S^n$. Assuming the theorem, $\mathrm{Conf}(S^{n-1} \times [0,\delta]) \cong \mathrm{Map}(S^{n-1}, S^n)$. It's then easy to see directly that the map $S^n \to \mathrm{Map}(S^{n-1}, S^n)$ does not have homotopy fiber $\coprod_i (\mathrm{Conf}_i(D^n_{1-\delta})_{\Sigma_i}$. (For instance, in the case of $S^1 \to \mathrm{Map}(S^0, S^1)$, the homotopy fiber ΩS^1 is homotopy equivalent to $\mathbb{Z}$.)

Now, let's just prove directly that our map isn't a fibration by presenting a lifting problem that can't be solved. Choose

$$\begin{tikzcd}
* \arrow[r, "\text{base}", "\text{point}"'] \arrow[d] & \mathrm{Conf}(D^n) \arrow[d] \\
{[0,1]} \arrow[r, "\gamma"] & \mathrm{Conf}(S^{n-1} \times [0,\delta])
\end{tikzcd}$$

where γ is the red path shown in the picture below:



That is, the path γ from time $t = 0$ to time ϵ is just the empty configuration. At time $t = \epsilon$, a point is created on the interior boundary of the annulus and then this point moves around until time $t = 1$ (what it does doesn't matter). We choose the lifting condition at time $t = 0$ of the empty configuration in $\mathrm{Conf}(D^n)$. It is impossible to lift the path γ, because of this initial condition and the fact that we cannot create points in D^n in its interior, only on its boundary. Thus, this map cannot be a fibration.

Note that we were unable to construct the lift of γ because the boundary of ∂U surjected onto the boundary ∂V. If this was not the case (e.g., if U looked like a horseshoe in V) then we could have created a point on the complement of ∂V in ∂U, then rushed, in time ϵ, the point along a path ending at $\gamma(\epsilon)$ on the interior boundary of U. This would have defined a lift. So, one should still feel like we're in good shape so long as the boundary of U doesn't surject onto the boundary of V. This is true, for instance, for the inclusions $S^{k-1} \times [0, \delta] \times D^{n-k} \hookrightarrow D^k \times D^{n-k}$, for $n > k$. We will continue with this issue next time.

References

[1] Bödigheimer, C.-F. Stable splittings of mapping spaces. Algebraic topology (Seattle, Wash., 1985), 174187, Lecture Notes in Math., 1286, Springer, Berlin, 1987.

[2] McDuff, Dusa. Configuration spaces. K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), pp. 88–95. Lecture Notes in Math.,#Vol. 575, Springer, Berlin, 1977.