THE H-PRINCIPLE, LECTURES 1 AND 2: OVERVIEW

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This class is about the h-principle. This topic might not have the huge name recognition for you all, which might lead you might think that the h-principle an esoteric or niche topic in mathematics, but it's not. It's a central and useful tool in geometry, and part of the impetus to teach this course is that it's easy to imagine you all making use of the h-principle in your own work.

1. Immersion theory

Given two smooth manifolds M and N, an interesting problem is to study smooth maps of M to N. A particularly interesting type of map (if $\dim(N) \geq \dim(M)$) is an embedding, i.e., a smooth math $M \to N$ which is also an injection at the level of sets. $\mathrm{Emb}(M, N)$ is a very interesting thing to try to understand; for instance, the study of $\mathrm{Emb}(S^1, \mathbb{R}^3)$ is knot thory. Studying embeddings is difficult in part because the property of a map f being an embedding is not local on M. There is a closely related type of map that is local on M, which is an immersion. Recall that $f : M \to N$ is an immersion if for every point $x \in M$, the derivative map on tangent spaces $df_x : T_xM \to T_{f(x)}N$ is injective. (Equivalently, f is an immersion if it is locally an embedding, i.e., every point x has an open neighborhood U such that $f|_U$ is an embedding.)

In the case where the source M is the standard m-disk D^m, there is a fairly simple model for both the spaces of embeddings and immersions into a target N. Given any immersion $f : D^m \to N$, we can differentiate at the origin to obtain an injection $df_0 : \mathbb{R}^m \cong T_0D^m \hookrightarrow T_{f(0)}N$. This is equivalent to a choice of an m-dimensional subspace of the tangent space TN at the point $f(0)$ together with a basis, which is a point of $V_m(TN)$, the Stiefel bundle of framed m-framed subspaces of the tangent bundle of N. Thus, we obtain a natural map $\mathrm{Emb}(D^m, N) \to \mathrm{Imm}(D^m, N) \to V_m(TN)$. Likewise, any m-framed subspace $\mathbb{R}^m \hookrightarrow T_xN$ can be used to define an embedding $\mathbb{R}^m \to N$ by using the exponential map $T_xN \to N$ (perhaps combined with a scaling). It is not hard to construct an isotopy between the embedding f and the embedding $\exp \circ df_0$, and this implies that the big, seemingly complicated, spaces $\mathrm{Emb}(D^m, N)$ and $\mathrm{Imm}(D^m, N)$ are both homotopy equivalent to a less intimidating bundle of frames $V_m(TN)$.

Since immersions are local in the source, one might then hope that this fairly simple model of immersions for source D^m should globalize to a similarly nice model. Of course, we made the arbitrary choice of a point $0 \in D^m$, so to globalize we should first fix this model so as to not depend on such a choice. The fix is called a formal immersion.

Definition 1.1. A formal immersion F from M to N is an injective bundle map $TM \to TN$. That is, F consists of a map $f : M \to N$ and a vector bundle map $F : TM \to TN$ covering f such that the map $F|_x : T_xM \to T_{f(x)}N$ is injective for every point x in M. $\mathrm{Imm}^{\mathrm{f}}(M, N)$ is the space of formal immersions.

This definition might look very similar to a basic notion in differential equations, that of a formal solution: If you have some differential equation, say $G(x, \dot{x}) = 0$, you can replace the derivative $\dot{x}$ with an independent variable, y, and try to first solve the equation $G(x, y) = 0$ as a stepping stone to finding genuine solutions of $G(x, \dot{x}) = 0$. Such an intermediate solution is sometimes called a formal solution.

There is a natural map $\mathrm{Imm}(M, N) \to \mathrm{Imm}^{\mathrm{f}}(M, N)$, sending an immersion f to the injective bundle map df. It might seem like the space $\mathrm{Imm}^{\mathrm{f}}(M, N)$ is more complicated than $\mathrm{Imm}(M, N)$,

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since we are allowing *more* maps (for instance, maps $TM \to TN$ which may not arise as the derivative of any immersion of M into N), but from the point of view of homotopy theory or cohomology, $\mathrm{Imm}^{\mathrm{f}}(M,N)$ can be analyzed much more easily than $\mathrm{Imm}(M,N)$, because of the existence of a fibering

$$\begin{tikzcd} \mathrm{Imm}^{\mathrm{f}}(M,N) \arrow[d] & \mathrm{Hom}^{\mathrm{inj}}_{\mathrm{Vect}_M}(TM, f^*TN) \arrow[l] \arrow[d] \\ \mathrm{Map}(M,N) & \{f\} \arrow[l] \end{tikzcd}$$

where the base and the fiber are often fairly comprehensible objects in homotopy theory.

For instance, it is a difficult theorem to show that the space $\mathrm{Imm}(M^m, \mathbb{R}^{2m})$ is nonempty (i.e., that every m-manifold immerses into $\mathbb{R}^{2m}$, a weak version of Whitney's immersion theorem), but it is comparatively easy to show that the space of formal immersions $\mathrm{Imm}^{\mathrm{f}}(M, \mathbb{R}^{2m}) \simeq \mathrm{Hom}^{\mathrm{inj}}(TM, \underline{\mathbb{R}}^{2m})$ is nonempty: You just need to show that there always exists an m-dimensional bundle N_M on M such that the direct sum $T_M \oplus N_M$ is a trivial bundle, and you obtain a formal immersion $T_M \hookrightarrow T_M \oplus N_M \cong \underline{\mathbb{R}}^{2m}$.

The following celebrated theorem was proven by Hirsch, generalizing work of Smale. Recall that a manifold M is open if the complement of the boundary, $M - \partial M$, has no compact component.

Theorem 1.2 (Smale-Hirsch immersion theory)**.** *Let M be compact, and assume that either M is open or* $\dim(M) < \dim(N)$*. Then the map* $\mathrm{Imm}(M,N) \to \mathrm{Imm}^{\mathrm{f}}(M,N)$ *is a weak homotopy equivalence.*

This theorem is remarkable in several respects. First, it reduces a situation in analysis involving a partial differential relation (an immersion) to one in topology + algebra (a formal immersion). Secondly, it allows for extensive computation. For instance, let us see what this theorem implies in the case of immersing spheres into Euclidean space, which was Smale's original result. First, note that there is a map $\mathrm{Imm}^{\mathrm{f}}(S^k, \mathbb{R}^{n+k}) \to \mathrm{Map}(S^k, V_n(\mathbb{R}^{n+k}))$, which can be shown to be a homotopy equivalence. A corollary of this coupled with the above theorem is then the following (which predates the above result):

Theorem 1.3 (Smale)**.** *Isotopy classes of immersions of S^k into $\mathbb{R}^{n+k}$ are in bijection with $\pi_k V_n(\mathbb{R}^{n+k})$, the kth homotopy group of $V_k(\mathbb{R}^{n+k}) \cong \mathrm{GL}_{n+k}/\mathrm{GL}_n$, the Stiefel manifold of k-frames in $\mathbb{R}^{n+k}$.*

For instance, the group $\pi_2 V_2(\mathbb{R}^3)$ classifies immersions of S^2 into $\mathbb{R}^3$. The Stiefel manifold $V_2(\mathbb{R}^3)$ is diffeomorphic to SO_3, and a basic exercise from topology shows $\pi_2 \mathrm{SO}_3 = 0$. Consequently, all immersions of S^2 into $\mathbb{R}^3$ are isotopic. In particular, S^2 can be turned inside-out inside $\mathbb{R}^3$ by moving through a family of immersions. This was a very surprising result; Smale's thesis advisor, Raoul Bott, reportedly told him that the result couldn't be true.

Remark 1.4. Neat as it is to be able to easily construct formal immersions, the case of parallelizable manifolds (such as S^1, S^3, S^7) shows that the conditions of the theorem are necessary, because sometimes it is too easy to construct formal immersions. E.g., $\mathrm{Imm}(M^n, \mathbb{R}^n)$ is the empty set if M is compact without boundary, but there always exists a formal immersion $TM^n \to T\mathbb{R}^n$ if TM is a trivial bundle, which just maps M to a single point in $\mathbb{R}^n$.

We now have an instance in which solutions (immersions) are equivalent to formal solutions (formal immersions), which one might have otherwise supposed would never be true in any interesting situations. This begs the question: What is the general principle at work here, and when does it hold?

2. The h-principle for differential relations

Let M be a smooth n-manifold. Let E be a smooth fibration over M, $E \to M$. The space of k-jets of the bundle E, $J_k E$, is the sections of a bundle $E^{(k)}$ over M whose fiber $J_k E|_x$ at a point

$x \in M$ is the space of smooth sections of E in a neighborhood of x modulo the equivalence relation that $f \sim g$ if they agree to order k in a neighborhood of x (i.e., if the first k derivatives of $f - g$ vanish when restricted to some arbitrarily small $\mathbb{R}^n \cong U \subset M$ containing x). Note that there is a canonical map $j^{(k)} : \Gamma(E) \to J_k E$ from sections of E to k-jets of sections of E.

Definition 2.1. A differential relation $\mathcal{R}$ of order k is a subspace of $E^{(k)}$. The space of (holonomic) solutions $\mathrm{Sol}_{\mathcal{R}}(M)$ of $\mathcal{R}$ is the image $\Gamma(E)$ in $\Gamma(\mathcal{R})$, i.e., the sections of $\mathcal{R}$ which are k-jets of actual section of E.

Definition 2.2. A differential relation $\mathcal{R}$ adheres to the h-principle if the space of (holonomic) solutions is weakly homotopy equivalent to the space of formal solutions. In other words, the map $\mathrm{Sol}_{\mathcal{R}}(M) \to \Gamma(\mathcal{R})$ induces an isomorphism on homotopy groups (for every choice of basepoint).

Remark 2.3. Ignore this on first reading: It is also useful to have some weaker notions of the h-principle. Say that $\mathcal{R}$ adheres to the j-parametric h-principle if the map $\mathrm{Sol}_{\mathcal{R}}(M) \to \Gamma(\mathcal{R})$ is j-connective. Caution: Many authors use "the h-principle" to refer to what we shall refer to as the 0-parametric h-principle. Also, say that

Gromov developed three basic techniques for establishing the h-principle: convex integration, removal of singularities, and microflexible sheaves.

Using the third technique, of sheaves, Gromov gave the following simple criterion for establishing the h-principle for $\mathrm{Diff}(M)$-invariant differential relations on open manifolds. First, assume that there is an action of $\mathrm{Diff}(M)$ on E compatible with the action on M. $\mathcal{R}$ is $\mathrm{Diff}(M)$-invariant if $\mathcal{R}$ is mapped to itself under the corresponding action on $E^{(k)}$. Say that $\mathcal{R}$ is open if it is an open subspace of $E^{(k)}$.

Theorem 2.4 (Gromov)**.** *Let $\mathcal{R}$ be an open,* $\mathrm{Diff}(M)$*-invariant, differential relation on M. If M is an open n-manifold, then $\mathcal{R}$ satisfies the h-principle.*

Example 2.5. Consider the a trivial product bundle $E = M \times N$ over M, so sections of E are the same as maps from M to N. The bundle of 1-jets of maps, $E^{(1)}$, has a subspace $\mathcal{R}$ consisting of bundle maps which are injective on each individual tangent space. A section of $\mathcal{R}$ is then the same thing as a formal immersion. In this situation, Gromov's theorem thus specializes to Hirsch's theorem when M is open.

3. The h-principle for presheaves of spaces

Those inclined to greater generalization, topologists less analytically inclined, or those who have an example of interest that doesn't quite fit in the rubric of differential relations, might ask whether a version of this h-principle exists without the trappings of analysis at all. One might imagine doing this by just remembering the structure of solutions to a differential relation $\mathcal{R}$ possess, but forgetting about the differential relation itself.

The space of sections of a diffeomorphism invariant differential relation is a particular type of presheaf:

Definition 3.1. $\mathcal{U}_M$ is the category of open subsets of M, with morphisms given by inclusions. A presheaf (of spaces) $\mathcal{F}$ on M is a contravariant functor from $\mathcal{U}_M$ to the category of topological spaces, $\mathcal{F} \in \mathcal{P}(M) = \mathrm{Fun}(\mathcal{U}_M^{\mathrm{op}}, \mathrm{Spaces})$. $\mathcal{F}$ is isotopy invariant if for every isotopy equivalence $i : U \hookrightarrow V$ in $\mathcal{U}_M$, then the map $\mathcal{F}(V) \to \mathcal{F}(U)$ is a homotopy equivalence. $\mathcal{F}$ is equivariant given a compatible collection of actions of $\mathrm{Diff}(U)$ on $\mathcal{F}(U)$ for each U in M.

Recall that $f : U \to V$ is an isotopy equivalence if there exists an embedding $g : V \to U$ such that fg and gf are isotopic to id_V and id_U. By the isotopy extension theorem, this is equivalent to the existence of a diffeomorphism f of M such that restriction of f to U defines a diffeomorphism $f : U \cong V$. Thus, $\mathrm{Diff}(M)$-equivariance implies isotopy invariance.

Observe that holonomic solutions to a diffeomorphism invariant differential relation $\mathcal{R}$ always form a isotopy invariant presheaf. The value of $\mathrm{Sol}_{\mathcal{R}}(U) := \mathrm{Image}(\Gamma_U(E)) \cap \Gamma_U(\mathcal{R})$, the space of holonomic solutions of the restriction of $\mathcal{R}$ to U. Since solutions restrict, we obtain a presheaf structure on $\mathrm{Sol}_{\mathcal{R}}$. The isotopy extension theorem implies that if $\mathcal{R}$ is diffeomorphism invariant, then $\mathrm{Sol}_{\mathcal{R}}$ is isotopy invariant.

In order to formulate the h-principle for sheaves, we need some analogue of the formal solutions of a differential relation. A hint is given by the fact that formal solutions to $\mathcal{R}$, if $\mathcal{R}$ is $\mathrm{Diff}(M)$-invariant, are sections of a bundle on M. Given an equivariant presheaf $\mathcal{F}$ on M, we can construct a bundle on M by taking the diagonal quotient $V_n(TM) \times_{\mathrm{GL}_n} \mathcal{F}(\mathbb{R}^n)$, where $V_n(TM)$ is the frame bundle of M and $\mathcal{F}(\mathbb{R}^n)$ is the value of $\mathcal{F}$ on an arbitrary open $\mathbb{R}^n \to M$, both of which are acted on by GL_n.

Definition 3.2. Given an equivariant presheaf $\mathcal{F}$ on M, the linear approximation $\mathcal{F}'$ to $\mathcal{F}$ is the sheaf of the sections of the fiber bundle $V_n(TM) \times_{\mathrm{GL}_n} \mathcal{F}(\mathbb{R}^n)$, $\mathcal{F}'(U) := \Gamma(V_n(TU) \times_{\mathrm{GL}_n} \mathcal{F}(\mathbb{R}^n))$.

One can construct natural map of presheaves $\mathcal{F} \to \mathcal{F}'$, called the scanning map after Segal [16].

Definition 3.3. An equivariant presheaf $\mathcal{F}$ on M satisfies the h-principle if the scanning map $\mathcal{F} \to \mathcal{F}'$ is a weak equivalence, i.e., if $\mathcal{F}(U) \to \mathcal{F}'(U)$ is a weak equivalence of spaces for every $U \subset M$.

Remark 3.4. The presheaf satisfying the h-principle is a type of homotopy sheaf condition. For instance, If $\mathcal{F}$ is a sheaf of spaces, and every restriction map $\mathcal{F}(V) \to \mathcal{F}(U)$ is a fibration, then $\mathcal{F}$ adheres to the h-principle. (Those familiar with model categories may recognize this as close to the fibrancy condition for the Joyal or Jardine model structure on presheaves of spaces.)

The following is list of interesting presheaves, most, but not all, of which adhere to the h-principle.[1] (I call them presheaves, but most them also satisfy the usual sheaf condition.)

- **Functions**: Let $\mathcal{O}$ be the presheaf of functions on a manifold M. $\mathcal{O}$ satisfies the h-principle.
- **Holomorphic functions**: Let $\mathcal{O}^{\mathrm{hol}}$ be the presheaf of holomorphic functions on a complex manifold M. Then $\mathcal{O}^{\mathrm{hol}}$ adheres to the h-principle if M is Stein, but $\mathcal{O}^{\mathrm{hol}}$ **fails** to satisfy the h-principle if M is compact.
- **Vector bundles**: Let Vect_n be the presheaf of n-dimensional vector bundles on M. Then Vect_n satisfies the h-principle.
- **Thom Transversality**: Let N be a smooth manifold with a submanifold $K \subset N$. Let $\mathrm{Map}_{\pitchfork K}(-, N)$ be the presheaf on M of maps to N that intersect K transversally. Thom's transversality theorem is equivalent to the statement that $\mathrm{Map}_{\pitchfork K}(-, N)$ satisfies the 0-parametric (and C^0-dense) h-principle.
- **Configuration spaces**: Let M be a compact manifold with nonempty boundary. Let $\tilde{C}$ be the presheaf on M assigning to U the unordered configuration space of all points in $\overline{U}$, with the equivalence relation that two configurations agree if they agree minus the points on the boundary, so $\tilde{C}(M) = (\coprod_{i \geq 1} \mathrm{Conf}_i(M))/\sim$, where $r \sim s$ if $r \cap (M - \partial M) = s \cap (M - \partial M)$. Then $\tilde{C}$ satisfies the h-principle, giving h-principle proofs of theorems of McDuff and Barratt-Quillen-Priddy.
- **Foliations**: Let $\mathcal{F}$ be the presheaf of codimension k foliations on M. Then a theorem of Haefliger implies that $\mathcal{F}$ adheres to the h-principle if M is open, and a theorem of Thurston implies it for general M.
- **Cobordisms**: Let Cob_d be the presheaf of cobordisms in M: $\mathrm{Cob}_d(U)$ is the classifying space of the category whose objects are d-dimensional submanifolds of U, and whose morphisms are cobordisms in $U \times [0, 1]$. Then Cob_d satisifies the h-principle, which gives a proof the Galatius-Madsen-Tilman-Weiss theorem on the homotopy type of the cobordism category.
- **Embeddings**: Let $\mathcal{E}_N$ be the presheaf of embeddings of subspaces of M into N, $\mathcal{E}_N(U) = \mathrm{Emb}(U, N)$. For the case of M being the n-disk, then $\mathcal{E}_N$ satisfies the h-principle. For

[1]Not a complete list.

essentially all other M, $\mathcal{E}_N$ does **not** obey the h-principle. (Note also that the $\mathcal{E}_N$ cannot be presented as the solutions to any differential relation $\mathcal{R}$.)

- **Submersions**: Let Subm_N be the presheaf of submersions of open subspaces of M onto N. Phillips-Gromov submersion theory says that Subm_N satisfies the h-principle if M is open.
- **Smoothing theory**: Let $\mathcal{S}$ be the presheaf of smooth structures on M. That is, for any open subset $U \subset M$, $\mathcal{S}(U)$ is the classifying space of the category of smooth manifolds with a homeomorphism to U. Then the smoothing theory of Kirby-Siebenmann implies that $\mathcal{S}$ satisfies the h-principle if the dimension of M is greater than 4. In contrast there is an uncountable set of smooth structures on $\mathbb{R}^4$, and this implies that $\mathcal{S}$ does not obey the h-principle if M has dimension 4.
- **Isometries**: For M^n a Riemannian manifold, $\mathbb{R}^q$ usual Euclidean space of dimension greater than n, let $\mathrm{Isom}^{C^1}_{\mathbb{R}^q}$ be the presheaf of isometric C^1-immersions of open subspaces of M into $\mathbb{R}^q$. The Nash-Kuiper theorem is roughly equivalent to the statement that $\mathrm{Isom}^{C^1}_{\mathbb{R}^q}$ satisfies the h-principle. (This implies, in particular, Nash's embedding theorem, the amazing result that every Riemannian manifold M isometrically embeds into Euclidean space.)
- **Symplectic forms**: Let M be an open almost complex manifold, and let $\mathcal{S}$ be the presheaf of compatible symplectic structures on M. Then $\mathcal{S}$ adheres to the h-principle, one of Gromov's first applications of the h-principle in symplectic geometry, providing plentiful symplectic structures on open manifolds.
- **Curvature**: Let $\mathrm{Sec}^{\mathrm{pos}}$ and $\mathrm{Sec}^{\mathrm{neg}}$ be the presheaves on M of Riemannian metrics of positive and negative sectional curvature, respectively. Then both $\mathrm{Sec}^{\mathrm{pos}}$ and $\mathrm{Sec}^{\mathrm{neg}}$ satisfy the h-principle.

Understanding some of these examples will be one of our main points of focus in this course.

4. Calculus of functors

What about when the h-principle fails, such as in the case of embeddings? Can the methods of linear approximation be extended to address this situation? One such methodology is provided by Goodwillie-Weiss's calculus of presheaves on manifolds.

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