THE H-PRINCIPLE, LECTURE 22: THE GOODWILLIE-WEISS CALCULUS OF PRESHEAVES ON MANIFOLDS

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Definition 0.1. A functor $\mathcal{F}\colon \mathrm{Mfld}_n^{\mathrm{op}} \to \mathrm{Spaces}$ is a *polynomial functor of degree* $\leq k$ if it satisfy the following for any finite set J of cardinality $|J| > k$. Suppose given a manifold $V \in \mathrm{Mfld}_n$ and its open submanifolds A_i indexed by $i \in J$, where A_i are the interior of compact manifolds whose disjoint union is embedded in V. Then the map

$$\mathcal{F}(V) \longrightarrow \operatorname{holim}_{\substack{S\subset J\\ S\neq\varnothing}} \mathcal{F}\left(V - \bigcup_{i\in S} A_i\right)$$

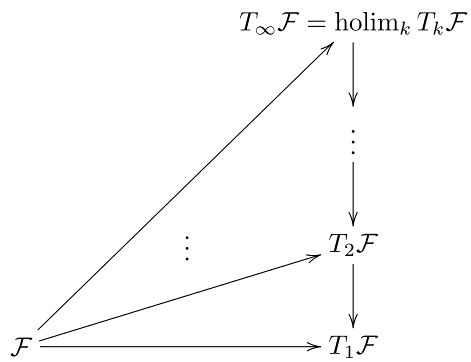
is a weak homotopy equivalence.

Remark 0.2. If $\mathcal{F}$ satisfies that $\mathcal{F}(\varnothing) \simeq *$, then it is a polynomial functor of degree ≤ 1 if and only in it is linear, i.e., it adheres to the h-principle.

Example 0.3. The functor $\mathcal{F}$ defined by $\mathcal{F}(M) = \mathrm{Map}(M^k, X)$ is a polynomial of degree k.

We shall define as follows, and then shall give a construction of it.

Definition 0.4. The *Taylor tower* of a functor $\mathcal{F}$ is a tower of functors



such that $T_k\mathcal{F}$ is a polynomial functor of degree $\leq k$ which is universal among polynomial functors of degree $\leq k$ receiving a map from $\mathcal{F}$.

Analogy of this with the calculus of functions is as in the following table.

functions	functors
f	$\mathcal{F}$
Taylor series	Taylor tower
$T_k f$, the k-th Taylor polynomial	$T_k\mathcal{F}$, the k-th Taylor polynomial approximation to $\mathcal{F}$

Next, we give a construction of $T_k\mathcal{F}$.

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Definition 0.5. For a manifold $M \in \mathrm{Mfld}_n$, let $\mathcal{U}(M)$ be the category of embedded codimension-0 submanifolds of M, and let $D_{\leq k}(M) \subset \mathcal{U}(M)$ be the category of embedded submanifolds $V \subset M$ such that $V \cong \coprod_{j \in J} D_j^n$ the disjoint union of n-dimensional disks, where $|J| \leq k$ (so

$$D_{\leq 0} \subset \cdots \subset D_{\leq k} \subset D_{\leq k+1} \subset \cdots \subset \mathcal{U}(M)).$$

Recall that another way of saying that $\mathcal{F}\colon \mathrm{Mfld}_n^{\mathrm{op}} \to \mathrm{Spaces}$ is linear is that the map

$$\mathcal{F}(M) \longrightarrow \mathrm{holim}_{D^n \subset M}\, \mathcal{F}(D^n),$$

where the limit is over all embedded n-disks, is a weak homotopy equivalence, since for any fibration $E \to M$, we have

$$\Gamma(M, E) \xrightarrow{\cong} \lim \Gamma(D^n, E),$$

and $\lim \Gamma(D^n, E) \simeq \mathrm{holim}\, \Gamma(D^n, E)$ since $\Gamma(-, E)$ sends all inclusions to fibrations. We can write this weak equivalence as

$$\mathcal{F}(M) \simeq \mathrm{holim}_{D \in D_{\leq 1}(M)}\, \mathcal{F}(D).$$

Definition 0.6. $T_k\mathcal{F}$ is the functor with values

$$T_k\mathcal{F}(M) = \mathrm{holim}_{V \in D_{\leq k}(M)}\, \mathcal{F}(V).$$

Equivalently, $T_k\mathcal{F} = (i_k)_* i_k^* \mathcal{F}$, where $i_k\colon D_{\leq k}(M) \hookrightarrow \mathcal{U}(M)$ is the inclusion, and $(i_k)_*$ is the (homotopy) right adjoint of the restriction functor

$$i_k^*\colon \mathrm{Fun}(\mathcal{U}(M)^{\mathrm{op}}, \mathrm{Spaces}) \to \mathrm{Fun}(D_{\leq k}(M)^{\mathrm{op}}, \mathrm{Spaces}).$$

That is, $T_k\mathcal{F}$ is the homotopy right Kan extension of $i_k^*\mathcal{F}$. The unit of the adjunction

$$\mathcal{F} \longrightarrow (i_k)_* i_k^* \mathcal{F} = T_k\mathcal{F}$$

gives a map from $\mathcal{F}$. The inclusion $D_{\leq k} \hookrightarrow D_{\leq k+1}$ gives rise to a diagram

$$\begin{array}{ccc} T_{k+1}\mathcal{F} & \longrightarrow & T_k\mathcal{F} \\ & \nwarrow \quad \nearrow & \\ & \mathcal{F} & \end{array}$$

commuting up to a canonical homotopy. This gives us a tower of functors under $\mathcal{F}$.

The following illustrate a result of a computation of $T_k\mathcal{F}$. As mentioned earlier the functor $\mathcal{F} = \mathrm{Map}((\ )^k, X)$ is a polynomial functor of degree k. Its truncations will be as follows.

As notation, let $\Delta_\ell(M^k)$ be the subspace of M^k consisting of all points $(x_1, \ldots, x_k)$ such that the image of the map

$$\{1, \ldots, k\} \longrightarrow M, \quad i \mapsto x_i$$

consists at most of ℓ-points. Concretely,

$$\Delta_1(M^k) \cong M \quad \text{(sitting as the diagonal in } M^k\text{)},$$
$$\Delta_\ell(M^k) = M^k \qquad \text{for all } \ell \geq k,$$
$$\Delta_{k-1}(M^k) = \{(x_1, \ldots, x_k) \mid x_i = x_j \text{ for some } i \neq j\},$$

and so on, and we have a sequence of subspaces.

$$\Delta_1(M^k) \subset \cdots \subset \Delta_k(M^k).$$

Corresponding to this, $T_i\mathcal{F}(M) \simeq \mathrm{Map}(\Delta_i(M^k), X)$ as towers of functors.

References

[1] Munson, Brian. Introduction to the manifold calculus of Goodwillie-Weiss. `http://arxiv.org/abs/1005.1698`
[2] Weiss, Michael. Embeddings from the point of view of immersion theory. I. Geom. Topol. 3 (1999), 67101.